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OPERATORS THAT ARE POINTS OF JOINT SPECTRAL COUTINUITY

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Introduction

Many results referring to the relation between numerical range $W(A)$ and spectrum $\sigma(A)$ of a single operator A have been investigated by many authors (Barra [3], Bonsall-Duncan [5], Hildebrandt [30], Maccluer [39], Meng [40, 41], Shiu [45, 46], Stampfli-Williams [49], Williams [53]). Especially after Newburgh ([43], 1949) studied upper semi-continuity and lower semi-continuity of set-valued functions, functions, continuity of spectra and numerical range have been tremendously developed for a long time (Bezak-Eisen [4], Conway-Morrel [13, 14, 15], Herrero [29], Luecke [35, 36, 37, 38], Murphy [42]). As much of knowledge in a single operator has been carried to the analogous situation in the case of n-tuple of operators, it would be quite reasonable to try to study the relation between joint spectrum and joint numerical range. Since its introduction by Arens-Calderon [2] the theory of joint spectra for commuting operators in a Hilbert space has recently been studied (Bunce [6], Chō-Takaguchi [8, 10], Coburn-Schechter [12], Curto [16, 17, 18, 19, 20], Harte [28], Patel [44], Slodkowski [47] and Vasilescu [51, 52]). Particularly, Abramov [1], Buoni [7], Chō-Takaguchi [9, 11], Danrun [21], Dash [22, 23, 24], Fillmore [25], Hildebrandt [31], Juneja [32], Taylor [50] and Zelazko [54] have investigated the relation between joint spectra and joint numerical range. Hence it would be reasonable to research what it means for joint spectra and joint numerical range to be continuous. The purpose of this paper is to discuss the continuity of these two functions. In this paper we use the sequential definition and the metric definition to show that the joint spectrum is continuous for an n-tuple of mutually commuting normal operators. As corollary we investigate the continuity of joint spectrum of an n-tuple of analytic Toeplitz operators in connection with the continuity of joint numerical range.

1. Continuity of joint spectrum

Though the notion of joint spectrum of a family of elements in a commutative Banach algebra was first introduced by Arens and Calderon [2], our interest here is the definition of joint spectra for an n-tuple of bounded operators on a Hilbert space which has been given by Harte and others [27, 48]. Let C^n be the n-dimensional complex space, $A=(A_1, A_2, \cdots, A_n)$ an n-tuple of commuting operators on a complex Hilbert space H with scalar product $\langle\ ,\ \rangle$ and associated norm, and $B(H)$ the Banach algebra of bounded linear operators on H, then the double commutant $\mathscr{U}$ of the set $S=\{A_1, A_2, \cdots, A_n\}$ is a weakly closed abelian algebra containing S and the identity.

DEFINITION 1. Let $A=(A_1, A_2, \cdots, A_n)$ be an n-tuple of commuting bounded linear operators on H, then the point $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ of C^n is *in the joint spectrum* $\sigma_J(A)$ of A relative to $\mathscr{U}$ if and only if for all $B_1, B_2, \cdots, B_n$ in $\mathscr{U}$, $B_1(A_1-\lambda_1)+B_2(A_2-\lambda_2)+\cdots+B_n(A_n-\lambda_n)\neq I$. Equivalently λ is in $\sigma_J(A)$ if and only if the ideal in $\mathscr{U}$ generated by $\{A_i-\lambda_i: 1\leq i\leq n\}$ is proper. Equivalently $(\lambda_1-A_1, \lambda_2-A_2, \cdots, \lambda_n-A_n)$ is singular.

It is well known, in fact, it can be easily shown that $\sigma_J(A)$ is non-empty and compact in C^n. That is, the joint spectrum is a function defined on $B(H)$ whose range consists of non-empty compact subsets of C^n.

EXAMPLE 1. In the case of a single operator A the definition of joint spectrum reduces to the usual definition of spectrum. The Koszul-complex for this case looks like $0\to H\xrightarrow[A-\lambda I]{} H\to 0$ and $A-\lambda I$ is non-singular if and only if $\ker(A-\lambda I)=\{0\}$ and $\mathrm{Im}(A-\lambda I)=H$ [33].

DEFINITION 2. For any n-tuple $A=(A_1, A_2, \cdots, A_n)$ of operators, the following non-negative numbers

$||A||_J=\sup\{(||A_1x||^2+||A_2x||^2+\cdots+||A_nx||^2)^{1/2}: ||x||=1\}$ and $r_J(A)=\sup\{(|\lambda_1|^2+|\lambda_2|^2+\cdots+|\lambda_n|^2)^{1/2}: \lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n))\in\sigma_J(A)\}$ are called ***the joint operator norm*** and ***the joint spectral radius*** respectively.

THEOREM 1. Joint spectrum is upper semi-continuous.

PROOF. First we prove by the metric definition.

Let K be the set of $A=(A_1, A_2, \cdots, A_n)$ where A_i $(i=1,2,\cdots,n)$ are singular operators on a fixed Hilbert space H, and given an n-tuple of operators $A=(A_1, A_2, \cdots, A_n)$, let $\phi(\lambda)$ be the distance from $A-\lambda$ to K. The function ϕ is continuous. If Λ_0 is an open set that includes joint spectrum $\sigma_J(A)$ of A, if Δ is the closed disc with center $0=(0,0,\cdots,0)$ and radius $1+\|A\|_J$, and if $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ $\in \Delta-\Lambda_0$, then $\phi(\lambda)>0$. (This does depend on K being closed; if $\phi(\lambda)=0$, i.e., $d(A-\lambda, K)=0$, then $A-\lambda \in K$, i.e., $\lambda \in \sigma_J(A)$.)

Since $\Delta-\Lambda_0$ is compact, there exists a positive number ε such that $\phi(\lambda)\geq\varepsilon$ for all λ in $\Delta-\Lambda_0$; there is clearly no loss of generality in assuming that $\varepsilon<1$. Suppose that $\|A-B\|_J<\varepsilon$ where $B=(B_1, B_2, \cdots, B_n)$. It follows that if $\lambda \in \Delta-\Lambda_0$, then $\|(A-\lambda)-(B-\lambda)\|_J<\varepsilon\leq d(A-\lambda, K)$. This implies that $B-\lambda$ is not in K, and hence that λ is not in $\sigma_J(B)$. Hence $\sigma_J(B)$ is disjoint from $\Delta-\Lambda_0$. At the same time, if $\lambda \in \sigma_J(B)$, then $|\lambda|\leq\|B\|_J\leq\|A\|_J+\|A-B\|_J$ $<1+\|A\|_J$, so that $\sigma_J(B)\subset\Delta$. These two properties of $\sigma_J(B)$ say exactly that $\sigma_J(B)\subset\Lambda_0$; hence the joint spectrum is upper semi-continuous. Next we prove by the sequential definition. Suppose that $A^m \longrightarrow A$ where $A^m=$ $(A_{1m}, A_{2m}, \cdots, A_{nm})$, $A=(A_1, A_2, \cdots, A_n)$, and $\mu=(\mu_1, \mu_2 \cdots, \mu_n) \in \overline{\lim_m}\ \sigma_J(A^m)$, choose $\lambda^m=(\lambda_{1m}, \lambda_{2m}, \cdots, \lambda_{nm})$ in $\sigma_J(A^m)$ so that for a suitable subsequence $\lambda^{m_k} \longrightarrow \mu$. Since $A_{im_k}-\lambda_{im_k}$ are singular, where $A^{m_k}=(A_{1m_k}, A_{2m_k}, \cdots, A_{nm_k})$, $\lambda^{m_k}=(\lambda_{1m_k}, \lambda_{2m_k}, \cdots, \lambda_{nm_k})$ and $A_{im_k}-\lambda_{im_k} \longrightarrow A_i-\mu_i (i=1,2,\cdots,n)$ and the mapping of the set of invertible operators onto itself defined by $A \longrightarrow A^{-1}$ is continuous, $A_i-\mu_i$ are singular, so that μ belongs to $\sigma_j(A)$. Hence the proof is complete.

In general the spectrum is not continuous. The following shift operator is an example. If $k=1,2,3\cdots$ and if $k=\infty$, let A_k be the two-sided weighted shift such that $A_k e_n$ is e_{n+1} or $(1/k)e_{n+1}$ according as $n\neq 0$ or $n=0$ (put $1/\infty=0$), then $\sigma(A_k)$ is not continuous. ([26] problem 102.)

THEOREM 2. The restriction of joint spectrum to the set of n-tuples $A=(A_1, A_2, \cdots A_n)$ of mutually commuting normal operators is continuous.

PROOF. To prove the statement, it is to be proved that if $\{A^m\}$ is a sequence of n-tuples of mutually commuting normal operators and $A^m\longrightarrow A$, then $\sigma_j(A)\subset \underline{\lim}_m\ \sigma_j(A^m)$. When is $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ not in $\underline{\lim}_m\,\sigma_j(A^m)$? Exactly when the distance from λ to $\sigma_j(A^m)$ does not tend to 0 as $n\to\infty$; in other words, exactly when there exists a positive number ε such that $d(\lambda,\sigma_j(A^m))\leq\varepsilon$ for infinitely many values of n. The inequality says that $1/d(\lambda,\lambda')\leq 1/\varepsilon$ whenever $\lambda'=(\lambda_1', \lambda_2', \cdots, \lambda_n')\in\sigma_j(A^m)$. This, in turn, says that not only is λ absent from $\sigma_j(A^m)$, so that $A_{im}-\lambda_i$ are invertible $(i=1,2,\cdots,n)$, but, in fact, $r_j((A^m-\lambda)^{-1})=\sup\{(|\mu_1|^2+|\mu_2|^2+\cdots\cdots+|\mu_n|^2)^{1/2}: \mu=(\mu_1, \mu_2, \cdots\cdots, \mu_n)\in\sigma_j((A^m-\lambda)^{-1})\}\leq\sup|\mu_1|+\sup|\mu_2|+\cdots\cdots+\sup|\mu_n|$, where $\mu_i\in\sigma((A_{im}-\lambda_i)^{-1})$. This equals to $r((A_{1m}-\lambda_1)^{-1})+r((A_{2m}-\lambda_2)^{-1})+\cdots\cdots+r((A_{nm}-\lambda_n)^{-1})=1/r(A_{1m}-\lambda_1)+1/r(A_{2m}-\lambda_2)+\cdots\cdots+1/r(A_{nm}-\lambda_n)\leq 1/d(\lambda,\lambda')\leq 1/\varepsilon$. Hence $r_j((A_m-\lambda)^{-1})\leq 1/\varepsilon$, where $r(A)=\sup\{|\lambda|: \lambda\in\sigma(A)\}$ is the spectral radius of a single operator A. Since for a commuting n-tuple of normal operators the joint spectral radius is the same as the joint operator norm [9], accordingly the contrapositive of what is to be proved is that if $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and a subsequence $\{A^{m_i}\}$ are such that $\|(A^{m_i}-\lambda)^{-1}\|_J \leq 1/\varepsilon$ for some ε, then $A_i-\lambda_i$ are invertible $(i=1,2,\cdots,n)$. There is no loss in simplifying the notation and assuming that $\|(A^m-\lambda)^{-1}\|_J \leq 1/\varepsilon$ for all m. Since $\|(A^m-\lambda)^{-1}-(A^l-\lambda)^{-1}\|_J=\|(A^m-\lambda)^{-1}((A^l-\lambda)-(A^m-\lambda))\cdot(A^l-\lambda)^{-1}\|_J \leq \|(A^m-\lambda)^{-1}\|_J\cdot\|A^l-A^m\|_J\cdot\|(A^l-\lambda)^{-1}\|_J \leq (1/\varepsilon^2)\|A^l-A^m\|_J$ and since $A^m\to A$, it follows that the sequence $\{(A^m-\lambda)^{-1}\}$ converges to some n-tuple of mutually commuting normal operator $B=(B_1, B_2, \cdots, B_n)$, say. Since $(A-\lambda)B=\lim_m (A^m-\lambda)\cdot\lim_m (A^m-\lambda)^{-1}=\lim_m (A^m-\lambda)\cdot(A^m-\lambda)^{-1}=I$. And similarly, of course, $B(A-\lambda)=I$. Hence the proof is complete.

DEFINITION 3. The *joint numerical range* $W_J(A)$ is defined as the set of all n-tuples of complex numbers $\{(\langle A_1x, x\rangle, \langle A_2x, x\rangle, \cdots, \langle A_nx, x\rangle) : \|x\|=1\}$.

In fact, $W_J(A)$ is a convex subset of $\boldsymbol{C}^n$. That is, the joint numerical range is a function defined on $B(H)$ whose range consists of convex subsets of $\boldsymbol{C}^n$.

Since the Hausdorff metric is defined for compact sets, the appropriate function to discuss is $cl(W_J(A))$, the closure of $W_J(A)$.

THEOREM 3. The function $cl(W_J(A))$ is continuous with respect to the uniform operator topology [34].

COROLLARY. If $A=(A_1, A_2, \cdots, A_n)$ is an n-tuple of analytic

Toeplitz operators, then convh $\sigma_J(A)$, the convex hull of $\sigma_J(A)$ is continuous.

PROOF. Convh $\sigma_J(A) = cl(W_J(A))$ [9].

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