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SOME PROPERTIES OF COMPLETELY POSITIVE MAP

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1. Introduction

In [1], Arveson stated that the correspondence between commutant of $\bar{\pi}(a)$ and the set of completely positive maps is an affine order isomorphism. This note states that the extension of B-valued inner product can not be carried out for even the simplest sort of pre-Hilbert B-module unless B is at least an AW^*-algebra [Theorem 2.9].

In § 3, in addition to Arveson's statements, it is also given that the correspondence preserves convex combinations [Theorem 3.5] and an equivalence condition for completely positive map [Theorem 3.6].

2. Preliminaries

DEFINITION 2.1. Let B be a C^*-algebra. *A pre-Hilbert B-module* is a right B-module X equipped with a conjugate bilinear map $\langle\ ,\ \rangle : X\times X\longrightarrow B$ satisfying:

(i) $\langle x,x\rangle \geq 0 \quad \forall x\in X$;

(ii) $\langle x,x\rangle = 0$ only if $x=0$;

(iii) $\langle x,y\rangle = \langle y,x\rangle^*$ for $x,y\in X$;

(iv) $\langle x\cdot b,y\rangle = \langle x,y\rangle b$ for $x,y\in X$, $b\in B$.

The map $\langle\ ,\ \rangle$ will be called a B-*valued inner product* on X.

EXAMPLE 2.2. If J is a right ideal of B, then J becomes a pre-Hibert B-module when we define $\langle\ ,\ \rangle$ by $\langle x, y\rangle = y^*x$ for $x, y \in J$.

For a pre-Hilbert B-module X, define $\|\cdot\|_X$ on X by $\|x\|_X = \|\langle x, x\rangle\|^{\frac{1}{2}}$.

PROPOSITION 2.3. $\|\cdot\|_X$ is a norm on X and satisfies:

(i) $\|x\cdot b\|_X \leq \|x\|_X \|b\|$ for $x \in X$, $b \in B$;

(ii) $\langle x, y\rangle^* \langle x, y\rangle \leq \|y\|^2{}_X \langle x, x\rangle$ for $x, y \in X$;

(iii) $\|\langle x, y\rangle\| \leq \|x\|_X \|y\|_X$ for $x, y \in X$.

PROOF. [5], [8].

DEFINITION 2.4. A pre-Hilbert B-module X which is complete with respect to $\|\cdot\|_X$ will be called a *Hilbert B-module.*

REMARK 2.5. For a pre-Hilbert B-module X, we let $\mathfrak{A}(X)$ denote the set of operators $T \in B(X)$ for which there is an operator $T^* \in B(X)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for $x, y \in X$. That is $\mathfrak{A}(X)$ is the set of bounded operators on X which possess bounded adjoint with respect to the B-valued inner product. It is easy to see that for $T \in \mathfrak{A}(X)$, the adjoint T^* is unique and belongs to $\mathfrak{A}(X)$, so $\mathfrak{A}(X)$ is a $*$-algebra with involution $T \to T^*$.

LEMMA 2.6. $\mathfrak{A}(X)$ consists of entirely module maps. i.e. if $T \in \mathfrak{A}(X)$, then $T(x\cdot b) = (Tx)\cdot b$ for $x \in X$, $b \in B$.

PROOF. Take $y \in X$. Then by properties of B-valued inner product,

$$\begin{aligned}\langle T(x\cdot b),\ y\rangle &= \langle x\cdot b,\ T^*y\rangle \\ &= \langle x,\ T^*y\rangle b \\ &= \langle Tx,\ y\rangle b = \langle (Tx)\cdot b, y\rangle.\end{aligned}$$

For the balance of this section, A will be a C^*-algebra, B a closed $*$-subalgebra of A, X a pre-Hilbert B-module, and Y a pre-Hilbert A-module.

LEMMA 2.7. For a linear map $T: X \longrightarrow Y$ the followings are equivalent:

(i) T is a bounded module map of B.

(ii) There is a real $K \geq 0$ such that $\langle Tx, Tx\rangle_A \leq K\langle x, x\rangle_B$ for $x \in X$.

PROOF. [1], [5].

We let X' denote the set of bounded B-module maps of X into B. By 2.7(with $A=B=Y$), X' is precisely the set of linear maps $\tau: X \longrightarrow B$ for which there is a real $K \geq 0$ such that $\tau(x)^*\tau(x) \leq K\langle x, x\rangle$ for $x \in X$. Each $x \in X$ gives rise to a map $\hat{x} \in X'$ defined by $\hat{x}(y) = \langle y, x\rangle$ for $y \in X$. We will call X *self-dual* if $\hat{X} = X'$. According to [5, p. 451], X' is a pre-Hilbert B-module, that is, $\langle\ ,\ \rangle$ can be extended to a B-valued inner product on X' and the extension satisfies $\langle \hat{x}, \tau\rangle = \tau(x)$ for $x \in X$ and $\tau \in X'$.

THEOREM 2.8. Let X and Y be pre-Hilbert A-modules and $T: X \longrightarrow Y$ a bounded module map. Then (i) There exists a bounded module map $\tilde{T}: X' \longrightarrow Y'$ (ii) $(\tilde{T}\hat{x})(y) = \widehat{(Tx)}(y)$ for $x \in X$ and $y \in Y$.

PROOF. (i) Define $T^{\#}: Y \longrightarrow X'$ by $(T^{\#}y)(x) = \langle Tx, y\rangle$

for $y \in X$, $x \in X$. By Schwarz's inquality $\|(T^{\#}y)(x)\| \leq \|T\|\|x\|\|y\|$, so $T^{\#}$ is bounded. Also since

$$\begin{aligned}(T^{\#}(y\cdot b))(x) = \langle Tx, y\cdot b\rangle = \langle y\cdot b, Tx\rangle^* \\ = (\langle y, Tx\rangle b)^* = b^*\langle y, Tx\rangle^* \\ = b^*\langle Tx, y\rangle = ((T^{\#}y)\cdot b)(x),\end{aligned}$$

$T^{\#}$ is a bounded module map.

Define $\tilde{T} : X' \longrightarrow Y'$ by $(\tilde{T}\tau)(y) = \langle T^{\#}y, \tau\rangle$ for $y \in Y$, $\tau \in X'$. Since $\tilde{T}$ is just $(T^{\#})^{\#}$, $\tilde{T}$ is a bounded module map also.

(ii) From the following observation, (ii) is immediate. That is, for $x \in X$, $y \in Y$,

$$\begin{aligned}(\tilde{T}\hat{x})(y) = \langle T^{\#}y, \hat{x}\rangle = \langle \hat{x}, T^{\#}y\rangle^* \\ = ((T^{\#}y)(x))^* = \langle Tx, y\rangle^* \\ = \langle y, Tx\rangle = \widehat{(Tx)}(y).\end{aligned}$$

THEOREM 2.9. Let B be a C^*-algebra with the property that for every right ideal J of B, there is a B-valued inner product $\langle\ ,\ \rangle$ on J' satisfying $\langle \hat{x}, \tau\rangle = \tau(x)$ for all $x \in J$, $\tau \in J'$. Then B is an AW^*-algebra.

PROOF. Let J be a right ideal of B. For $a \in B$, define $\tilde{a} \in J'$ by $\tilde{a}(x) = a^*x$ $(x \in J)$ and let $\tau_i \in J'$ denote the inclusion of J into B. Notice $\tau_i\cdot a = \tilde{a}$ for $a \in B$ and that $\tilde{x} = \hat{x}$ for $x \in J$.

Put $q = \langle \tau_i, \tau_i\rangle$. Then $q = q^*$ and $x \in J$, $qx = \tilde{q}(x)$. By the way,

$$\begin{aligned}\tilde{q}(x) = \langle \tau_i\cdot x, \tau_i\rangle = \langle \tilde{x}, \tau_i\rangle \\ = \langle \hat{x}, \tau_i\rangle = \tau_i(x) = x.\end{aligned}$$

So we have

$$q^2=\langle \tau_i\cdot q, \tau_i\rangle = \langle \tilde{q}, \tau_i\rangle = \langle \tau_i, \tau_i\rangle = q,$$

i.e., q is a projection. Put $p=1-q$, so p is a projection in $L(J)$, where $L(J)$ is the left annihilator of J. For self-adjoint element a of $L(J)$, we have $\tilde{a}=0$, so

$$qa=\langle \tau_i\cdot a, \tau_i\rangle = \langle \tilde{a}, \tau_i\rangle = 0,$$

which shows that $ap=a$ for all $a\in L(J)$. That is, $L(J)$ is a principal ideal generated by some projection p.

3. Completely positive maps

DEFINITION 3.1. Let B be a C^*-algebra, A a $*$-algebra and $\phi: A \longrightarrow B$ a linear map. We call ϕ *positive* if $\phi(a^*a)\geq 0$ $\forall a\in A$.

For $n=1,2,\cdots$, ϕ induces a map ϕ_n from algebra $A_{(n)}$ of $n\times n$ matrices with entries in A(made into a $*$-algebra by setting $[a_{ij}]^*=[a_{ji}^*]$ for matrices $[a_{ij}]\in A_{(n)}$) into the corresponding C^*-algebra $B_{(n)}$ defined by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$; we say that ϕ is *completely positive* if each of the induced map ϕ_n is positive.

REMARK 3.2. According to [7, p. 194], a linear map $\phi: A\longrightarrow B$ is completely positive if and only if

$$\sum_{i,j} b_i^*\phi(a_i^*a_j)b_j\geq 0 \text{ for } a_1,\cdots,a_n\in A,\ b_1,\cdots,b_n\in B.$$

Let ϕ be a completely positive and suppose in addition that $\phi(a^*)=\phi(a)^*$ for $a\in A$. The map ϕ give rise to a pre-Hilbert B-module as follows: consider the algebraic tensor product $A \otimes B$, which becomes a right B-module when we set $(a \otimes b) \cdot \beta = a \otimes b\beta$ for $b, \beta \in B$, $a \in A$.

Define

$$[\ ,\ ] : (A \otimes B) \times (A \otimes B) \longrightarrow B$$
$$\Big(\sum_{j=1}^{n} a_j \otimes b_j,\ \sum_{i=1}^{m} \alpha_i \otimes \beta_i\Big) \longrightarrow \Big[\sum_{j=1}^{n} a_j \otimes b_j,\ \sum_{i=1}^{m} \alpha_i \otimes B_i\Big]$$
$$= \sum_{i,j} \beta_i^* \phi(\alpha_i^* a_j) b_j,$$

for $a_1, \cdots a_n, \alpha_1, \cdots, \alpha_m \in A$, $b_1 \cdots b_n, \beta_1, \cdots, \beta_m \in B$.

$[\ ,\ ]$ is clearly well-defined and conjugate-bilinear. Since ϕ is completely positive, for with $x \in A \otimes B$, $[x, x] \geq 0$, since ϕ is $*$-map, $[x, y] = [y, x]^*$, and $[x \cdot b, y] = [x, y]b$ for $x, y \in A \otimes B$ and $b \in B$.

Put

$$N = \{x \in A \otimes B : [x, x] = 0\}.$$

Then N is a submodule of $A \otimes B$ and $X_0 = A \otimes B / N$ is a pre-Hilbert B-module with B-valued inner product

$$\langle x + N,\ y + N \rangle = [x, y] \quad \text{for } x, y \in A \otimes B.$$

Following T. W. Palmer [3], we call an element v of the $*$-algebra A *quasi-unitary* if $vv^* = v^*v = v + v^*$ and say that A is a *U^*-algebra* if it is the linear span by its quasi-unitary elements. All Banach $*$-algebras are U^*-algebra and A is a U^*-algebra iff it is spanned by its unitaries [3].

THEOREM 3.3. Let A be a U^*-algebra with 1, B a C^*-algebra with 1, and $\phi : A \longrightarrow B$ a completely positive map. Then

(i) there is a Hilbert B-module X, a $*$-representation π of A on X, and an element $e \in X$ such that $\phi(a) = \langle \pi(a)e, e \rangle$ for $a \in A$.

(ii) the set $\{\pi(a)(e\cdot b) : a \in A,\ b \in B\}$ spans a dense subspace of X.

PROOF. [5], [6], [7]. In particular, note that $\pi(a)(x+N) = a\cdot x+N\ \forall x \in A \otimes B$ and $\pi(a) \in \mathfrak{A}(X)$ (i.e., $\pi(a)$ is a B-module map), X a completion of X_0.

Let A be a U^*-algebra with 1, and B a C^*-algebra. If X, π and $e(e=1\otimes 1+N)$ are as in 3.3, we may define a $*$-representation $\tilde{\pi}$ of A on the self-dual Hilbert B-module X' by $\tilde{\pi}(a) = \pi(a)^{\sim} \in \mathfrak{A}(X')$ for $a \in A$. Suppose $\psi : A \longrightarrow B$ is another completely positive map. We write $\psi \leq \phi$ if $\phi-\psi$ is completely positive and let $[0, \phi]$ denote the set of completely positive maps from A into B which are $\leq \phi$.

For $T \in \mathfrak{A}(X')$, define $\phi_T : A \longrightarrow B$ by $\phi_T(a) = \langle T\tilde{\pi}(a)\hat{e}, \hat{e}\rangle$. Notice that $\phi_I = \phi$ and that the map $T \longrightarrow \phi_T$ is a linear map of $\mathfrak{A}(X')$ into the space of linear transformations of A into B, also that X' becomes a right B-module if we set $(\tau\cdot b)(x) = b^*\tau(x)$ for $\tau \in X'$, $b \in B$, $x \in X$.

LEMMA 3.4. Let A be a C^*-algebra with 1 and let $a \in A$, $a \geq 0$. Then there exists a unique element $b \in A$ such that $b \geq 0$ and $b^2 = a$.

PROOF. [2], [7].

THEOREM 3.5. Under the above circumstances,

(i) for each $T \in \tilde{\pi}(A)'$ with $0 \leq T \leq I_{X'}$, the formula $\phi_T(a) = \langle T\tilde{\pi}(a)\hat{e}, \hat{e}\rangle$ defines a completely positive map such that $\phi_T \leq \phi$;

(ii) the correspondence $T \longrightarrow \phi_T$ described in (i) is a bijection of $\{T \in \tilde{\pi}(A)' : 0 \leq T \leq I_{X'}\}$ onto $[0, \phi]$;

(iii) the correspondence preserves convex combinations, where $\tilde{\pi}(A)'$ denotes the commutant of $\pi(A)$ in $\mathfrak{A}(X')$.

PROOF. (i) For $a_1, \cdots, a_n \in A$ and $b_1, \cdots, b_n \in B$, set $x = \sum_{j=1}^{n} \pi(a_j)(e \cdot b_j) \in X$ (by 3.3, this is possible). Then

$$\begin{aligned}
\sum_{i,j} b_i^* \phi_T(a_i^* a_j) b_j &= \sum_{i,j} b_i^* \langle T\tilde{\pi}(a_i^* a_j)\hat{e}, \hat{e}\rangle b_j \\
&= \sum \langle T\tilde{\pi}(a_i^* a_j)\hat{e}\cdot b_j,\ \hat{e}\cdot b_i\rangle \\
&= \sum \langle T\tilde{\pi}(a_j)\hat{e}\cdot b_j, \tilde{\pi}(a_i)\hat{e}\cdot b_i\rangle \\
&\qquad (\text{since } T \in \tilde{\pi}(A)') \\
&= \langle T(\sum \tilde{\pi}(a_j)\hat{e}\cdot b_j),\ \sum \tilde{\pi}(a_i)\hat{e}\cdot b_i\rangle \\
&= \langle T\sum \tilde{\pi}(a_j)(e\cdot b_j)^\wedge,\ \sum \tilde{\pi}(a_i)(e\cdot b_i)^\wedge\rangle \\
&\qquad (\text{by the above notice}) \\
&= \langle T(\sum \tilde{\pi}(a_j)(e\cdot b_j))^\wedge,\ (\sum \pi(a_i)(e\cdot b_i))^\wedge\rangle \\
&\quad (\text{by } 2.8) \\
&= \langle T\hat{x}, \hat{x}\rangle \\
&= \langle T^{\frac{1}{2}}\hat{x},\ T^{\frac{1}{2}}\hat{x}\rangle \geq 0 \ (\text{by } 3.4).
\end{aligned}$$

Thus ϕ_T is completely prositive. But $\phi_T \leq \phi$ is to be shown in (ii).

(ii): If $T \in \tilde{\pi}(A)'$ and $\phi_T = 0$, then

$$\begin{aligned}
&\langle T(\pi(a_1)(e\cdot b))^\wedge,\ (\pi(a_2)(e\cdot b))^\wedge\rangle \\
&\qquad = \langle T\tilde{\pi}(a_1)(e\cdot b_1)^\wedge,\ \tilde{\pi}(a_2)(e\cdot b)^\wedge\rangle \\
&\qquad = \langle T\tilde{\pi}(a_2^* a_1)(e\cdot b_1)^\wedge,\ (e\cdot b_2)^\wedge\rangle \\
&\qquad = \langle T\tilde{\pi}(a_2^* a_1)\hat{e}\cdot b_1,\ \hat{e}\cdot b_2\rangle \\
&\qquad = b_2^* \langle T\tilde{\pi}(a_2^* a_1)\hat{e}, \hat{e}\rangle b_1 \\
&\qquad = b_2^* \phi_T(a_2^* a_1) b_1 = 0
\end{aligned}$$

for $a_1, a_2 \in A$, $b_1, b_2 \in B$.

So $\langle T(\hat{X}_0), X_0\rangle = 0$, or $\langle T(X), X\rangle = 0$; hence $T = 0$ by 2.8. Thus the correspondence is one-one.

To show that the conespondence is onto, take $\psi \in [0, \phi]$. By 3.3 there exists a $*$-representation ρ of A on a Hilbert

B-module Y and a $d \in Y$ such that $\phi(a) = \langle \rho(a)d, d \rangle$ for $a \in A$ and the set $\{\rho(a)(d \cdot b) : a \in A, \ b \in B\}$ spans a dense subspace Y_0 of Y. Since $\psi \leq \phi$, there is a well-defined bounded module map $W : X_0 \to Y_0$ such that $W(\pi(a)(e \cdot b)) = \rho(a)(d \cdot b)$ for $a \in A$, $b \in B$ and $\langle w_x, w_x \rangle \leq \langle x, x \rangle$ for $x \in X_0$. W extends to a bounded module map $W : X \longrightarrow Y$. Also

$$
\begin{aligned}
W\pi(a)(\pi(a)(e \cdot b)) &= w\pi(a)(a \otimes b + N) \\
&= w(a^2 \otimes b + N) \\
&= \rho(a^2)(d \cdot b) \\
&= \rho(a)\rho(a)(d \cdot b) = \rho(a) W(\pi(a)(e \cdot b)),
\end{aligned}
$$

i.e., $W\pi(a)$ and $\rho(a)w$ agree on X_0 for $a \in A$. Hence $W\pi(a) = \rho(a)W$ for $a \in A$. By 2.8, we get a bounded module map $\tilde{W} : X' \longrightarrow Y'$ extending W. It is clear from the proof of 2.8 that

$$\langle \tilde{W}\tau, \tilde{W}\tau \rangle \leq \langle \tau, \tau \rangle \quad \text{for } \tau \in X'.$$

Let $\tilde{W}^* : Y' \longrightarrow X'$ be the adjoint of $\tilde{W}$ and put $T = \tilde{W}^* W$, so $T \in \mathfrak{A}(X')$ and $T = T^*$. For $\tau \in X'$, we have $\langle T\tau, \tau \rangle = \langle \tilde{W}\tau, \tilde{W}\tau \rangle \leq \langle \tau, \tau \rangle$, so $0 \leq \langle T\tau, \tau \rangle \leq \langle \tau, \tau \rangle$, hence $0 \leq T \leq I$. Since $\tilde{W}\tilde{\pi}(a) = \tilde{\rho}(a)\tilde{W}$, $\tilde{W}(a)\tilde{W}^* = \tilde{W}^*\tilde{\rho}(a)$ for $a \in A$. Hence for any $a \in A$, we have

$$
\begin{aligned}
T\tilde{\pi}(a) &= \tilde{W}^*\tilde{W}\tilde{\pi}(a) \\
&= \tilde{W}^*\tilde{\rho}(a)\tilde{W} \\
&= \tilde{\pi}(a)\tilde{W}^*\tilde{W} = \tilde{\pi}(a)T, \quad \text{i.e., } T \in \tilde{\pi}(A)'.
\end{aligned}
$$

Finally, for $a \in A$,

$$
\begin{aligned}
\phi_T(a) = \langle T\tilde{\pi}(a)\hat{e}, \hat{e} \rangle &= \langle \tilde{W}\tilde{\pi}(a)\hat{e}, \tilde{W}\hat{e} \rangle \\
&= \langle W\pi(a)e, We \rangle \\
&= \langle \rho(a)d, d \rangle = \psi(a).
\end{aligned}
$$

These complete the proof of (i) and (ii).

(iii) The set $K = \{T \in \tilde{\pi}(A)' : 0 \leq T \leq I_{X'}\}$ is obviously convex; so is $[0, \phi]$. If $S, T \in K$ and $0 \leq \lambda \leq 1$, and $R = \lambda S + (1-\lambda)T$, then for $a \in A$,

$$\begin{aligned}\phi_R(a) &= \phi(a)\phi_{\lambda S+(1-\lambda)T}(a)\\ &= \langle(\lambda S+(1-\lambda)T)\tilde{\pi}(a)\hat{e},\ \hat{e}\rangle\\ &= \lambda\phi_S(a)+(1-\lambda)\phi_T(a),\end{aligned}$$

i.e., $\phi_{\lambda S+(1-\lambda)T} = \lambda\phi_S + (1-\lambda)\phi_T$.

Thus the correspondence preserves convex combinations.

THEOREM 3.6. A completely positive map ψ on A satisfies $\psi \leq \phi$ if and only if there exists an operator $T \in \mathfrak{A}(X')$ such that $0 \leq T \leq I_{X'}$, $T\tilde{\pi}(a) = \tilde{\pi}(a)T$ for all $a \in A$, and $\psi(a) = \langle T\tilde{\pi}(a)\hat{e}, \hat{e}\rangle$ for all $a \in A$.

PROOF. Suppose $\psi \in [0, \phi]$. Then by Theorem 3.5, $\psi(a) = \phi_T(a) = \langle T\tilde{\pi}(a)\hat{e}, \hat{e}\rangle$ and also by Theorem 3.5, it is clear.

Conversely, since $\phi_I = \phi$ and $\sum_{i,j} b_i^*(\phi - \phi_T)(a_i^* a_j)b_j = \langle(I-T)\hat{x},\ \hat{x}\rangle \geq 0$, for $a_1, \cdots, a_n \in A$, $b_1, \cdots, b_n \in B$ and $x = \sum_{j=1}^{n} \pi(a_j)(e\cdot b_j) \in X$, the proof is immediate.

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