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NONLINEAR ERGODIC THEOREMS FOR A NONEXPANSIVE SEMIGROUP IN UNIFORMLY CONVEX BANACH SPACES

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1. Introduction

Baillon ([1]) proved the first nonlinear ergodic theorem for nonexpansive mappings: Let C be a closed convex subset of a Hilbert space H and T a nonexpansive mapping of C in to itself. If the set $F(T)$ of fixed points of T is nonempty, then for $x\in C$, the Cesâro means

$$S_n x=\frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly as $n\to\infty$ to some $p\in F(T)$.

A corresponding result for a strongly continuous one parameter semigroup of nonexpansive mappings $S(t)$, $t\geq 0$ was proved soon after Baillon's work by Baillon and Brézis ([3]), i. e.,

$$A_\lambda x=\frac{1}{\lambda}\int_0^\lambda S(t)x dt$$

converges weakly as $\lambda\to\infty$ to a common fixed point of $S(t)$, $t\geq 0$.

These theorems were extended to Banach spaces by

Baillon ([2]), Bruck ([4]), Hirano ([7]), Reich ([11]), and Takahashi ([12]). By the way, above results are the cases for existence of weak limit of Cesàro means. From the example of Genel and Lindenstrauss ([6]), it follows that there exists a nonexpansive mapping such that the Cesàro mean does not converge strongly. Therefore, Pazy ([10]), Kim and Ha ([8]), and Kobayashi and Miyadera ([9]) give some further assumptions on the mapping in order to assure the strong convergence of the Cesàro means.

In this paper, we prove the existence of strong limit of the Cesàro means

$$A_t S(h)x = \frac{1}{t}\int_0^t S(s+h)x\,ds$$

uniformly in $h \geq 0$.

2. Preliminaries and notations

Let C be a closed convex subset of a uniformly convex Banach space X. A family $S=\{S(t): t \geq 0\}$ of mappings from C into itself is called a nonexpansive semigroup on C if

(1) $S(t+s)=S(t)S(s)$ for all t, $s \geq 0$,

(2) $S(0)=I$ (indentity),

(3) $\lim_{t\to 0+} S(t)x = x$ for every $x \in C$,

(4) $\|S(t)x - S(t)y\| \leq \|x-y\|$ for all $x, y \in C$ and $t \geq 0$.

The set of common fixed points of $S(t)$, $t \geq 0$ will be denoted by $F(S)=\bigcap_{t\geq 0} F(S(t))$. The Cesàro mean of $S(t)$,

$t \geq 0$ will be denoted by

$$A_\lambda x = \frac{1}{\lambda}\int_0^\lambda S(t)x dt.$$

It is easy to see that

$$A_t S(h)x = \frac{1}{t}\int_0^t \left\{ \frac{1}{s}\int_0^s S(h+\xi+\eta)x d\eta \right\} d\xi$$
$$+\frac{1}{ts}\int_0^s (s-\eta)\{S(h+\eta)x - S(h+t+\eta)x\}d\eta$$

for t, $s > 0$ and $h \geq 0$.

3. Main results

Now, we start with the following crucial lemmas to prove Theorem 3.4.

LEMMA 3.1. Let C be a closed convex subset of a uniformly convex Banach space X and $\{(t): t \geqq 0\}$ a nonexpansive semigroup on C. Suppose that $\lim_{t\to 0} \|S(t)x - S(t+i)x\|$ exists uniformly in $i > 0$. Then we have

$$\lim_{s,t\to\infty}\Big\|\frac{1}{2}(A_t S(t+h)x + A_s S(s+h)x)$$
$$-S(h)\Big(\frac{1}{2}A_t S(t)x + \frac{1}{2}A_s S(s)\, x\Big)\Big\| = 0$$

uniformly in $h>0$. In particular,

$$\lim_{t\to\infty}\|A_t S(t+h)x - S(h)A_t S(t)x\| = 0$$

uniformly in $h > 0$.

PROOF. Let $f \in F(S)$ and $r > 0$ with $\|x-f\| \leqq r$. Define the set $D = \{z \in X: \|z-f\| \leqq r\} \subset C$ and $U(t) = S(t)|D$

, the restriction of $S(t)$ to D. Then D is bounded closed convex and $U(t)$ a nonexpansive semigroup on D. Hence, by Theorem 2.1 in [5] (cf. [4] Lemma 1.1.), there exists a strictly increasing, continuous convex function $\gamma: R^+ \longrightarrow R^+$ with $\gamma(0)=0$ such that

$$\left\| U(h)\left(\sum_{i=1}^{k} \lambda_i x_i\right) - \sum_{i=1}^{k} \lambda_i U(h) x_i \right\| \leqq \gamma^{-1}\left(\max_{1 \leq i, j \leq k} \{\|x_i - x_j\| - \|U(h)x_i - U(h)x_j\|\}\right)$$

for any $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_k \geqq 0$ with $\sum_{i=0}^{k} \lambda_i = 1$, any $x_1, x_2, x_3, \cdots, x_k \in D$ and any $k \geqq 1$, $h > 0$. Consequently

$$\begin{aligned} &\left\| S(h)\left(\sum_{i=0}^{n-1} \lambda_i x_i + \sum_{i=0}^{m-1} \mu_i y_i\right) - \left(\sum_{i=0}^{n-1} \lambda_i S(h) x_i + \sum_{i=0}^{m-1} \mu_i S(h) y_i\right) \right\| \\ &\leqq \gamma^{-1}(\max \{\|x_i - x_j\| - \|S(h)x_i - S(h)x_j\|, \\ &\qquad \|x_i - y_p\| - \|S(h)x_i - S(h)y_p\|, \\ &\qquad \|y_p - y_q\| - \|S(h)y_p - S(h)y_q\| : \\ &\qquad 0 \leqq i, j \leqq n-1, 0 \leqq p, q \leqq m-1\}) \end{aligned}$$

for any $\lambda_i, \mu_i \geqq 0$ with $\sum_{i=0}^{n-1} \lambda_i + \sum_{i=0}^{m-1} \mu_i = 1$, any $x_i, y_i \in D$ and $n, m \geqq 1$, $h > 0$. Using this inquality with $\lambda_i = \frac{1}{2t}$, $x_i = S(t+i)x$, $\mu_i = \frac{1}{2s}$, $y_i = S(s+i)x$, and if we shall have the integral with respect to i instead of the summation. Then we have

$$\begin{aligned} &\left\| S(h)\left\{\int_0^t \frac{1}{2t} S(t+i)x di + \int_0^s \frac{1}{2s} S(s+i)x di\right\} \right. \\ &\left. \quad - \left\{\int_0^t \frac{1}{2t} S(h+t+i)x di + \int_0^s \frac{1}{2s} S(h+si)x di\right\} \right\| \end{aligned}$$

$$\begin{aligned}\leqq \gamma^{-1}(\max\{&\|S(t+i)x-S(t+j)x\|\\ &-\|S(h+t+i)x-S(h+t+j)x\|,\\ &\|S(t+i)x-S(s+p)x\|\\ &-\|S(h+t+i)x-S(h+s+p)x\|,\\ &\|S(s+p)x-S(s+q)x\|\\ &-\|S(h+s+p)x-S(h+s+q)x\|:\\ &0\leqq i,j\leqq t,\quad 0\leqq p,q\leqq s\})\end{aligned}$$

for any $s, t \geqq 0$ and $h > 0$.

For any $\varepsilon > 0$ choose $\delta > 0$ such that $\gamma^{-1}(\delta) < \varepsilon$. Since $\lim_{t\to\infty} \|S(t)x-S(t+i)x\|$ exists uniformly in $i > 0$, there exists $t_0 \geqq 0$ such that

$$\beta(i) \leqq \|S(t)x-S(t+i)x\| < \beta(i)+\delta$$

for every $i > 0$ and $t \geqq t_0$, where $\beta(i) = \lim_{t\to\infty} \|S(t)-S(t+i)x\|$. Hence if $s, t \geqq t_0$ then

$$\begin{aligned}&\|S(t+i)x-S(s+j)x\|-\|S(h+t+i)x-S(h+s+j)x\|\\ &\quad< \beta(|s+j-t-i|)+\delta-\beta(|s+j-t-i|)=\delta\end{aligned}$$

for all $i, j \geqq 0$. Consequently, we obtain that if $s, t \geqq t_0$ then

$$\begin{aligned}&\Big\|S(h)\Big\{\int_0^t \frac{1}{2t} S(t+i)xdi+\int_0^s \frac{1}{2s} S(s+i)xdi\Big\}\\ &\quad-\Big\{\int_0^t \frac{1}{2t} S(h+t+i)xdi+\int_0^s \frac{1}{2s} S(h+s+i)xdi\Big\}\Big\|\\ &\leqq \gamma^{-1}(\delta) < \varepsilon\end{aligned}$$

for any $h > 0$. Therefore, the result holds true. Furthermore, putting $s = t$, we have

$$\lim_{t\to\infty} ||A_t S(t+h)x - S(h)A_t S(t)x|| = 0$$

uniformly in $h>0$.

LEMMA 3.2. Let X, C and $\{S(t): t \geqq 0\}$ be as in Lemma 3.1. Let $x \in C$ and $F(S) \neq \phi$. If

$$\lim_{t\to\infty} ||S(t)x - S(t+i)x||$$

exists uniformly in $i>0$, then $\{||A_t S(t)x - f||\}$ is convergent for every $f \in F(S)$.

PROOF. Let $f \in F(S)$ and

$$\alpha_t = \sup_{h \geqq 0} ||A_t S(t+h)x - S(h)A_t S(t)x||$$

for $t \geqq 0$. Since

$$\begin{aligned}
A_{t+s}S(t+s)x &= \frac{1}{t+s}\int_0^{t+s} \{A_t S(s+t+\eta)x - S(s+\eta)A_t S(t)x\}d\eta \\
&\quad + \frac{1}{t+s}\int_0^{t+s} S(s+\eta)A_t S(t)x d\eta \\
&\quad + \frac{1}{t(t+s)}\int_0^t (t-\eta)\{S(s+t+\eta)x - S(2(t+s)+\eta)\, x\}d\eta, \\
||A_{t+s}S(t+s)x - f|| &\leqq \alpha_t + \frac{1}{t+s}\int_0^{t+s} ||S(s+\eta)A_t S(t)x - f||d\eta \\
&\quad + \frac{1}{t(t+s)}\int_0^t (t-\eta)||S(s+t+\eta)x - S(2(t+s)+\eta)x||d\eta \\
&\leqq \alpha_t + ||A_t S(t)x - f|| + \frac{t}{2(t+s)}||x - S(t+s)x|| \\
&\leqq \alpha_t + ||A_t S(t)x - f|| + \frac{t}{t+s}||x - f||
\end{aligned}$$

for all $t, s \geqq 0$. Letting $s \to \infty$, we have

$$\limsup_{s\to\infty} ||A_s S(s)x-f|| \leqq \alpha_t + ||A_t S(t)x-f||$$

for all $t \geqq 0$. Since $\lim_{t\to\infty} \alpha_t = 0$,

$$\limsup_{s\to\infty} ||A_s S(s)x-f|| \leqq \liminf_{t\to\infty} ||A_t S(t)x-f||.$$

Hence $\{||A_t S(t)x-f||\}$ is convergent for every $f \in F(S)$.

PROPOSITION 3.3. Let X, C and $\{S(t): t \geqq 0\}$ be as in Lemma 3.1. Let $x \in C$ and $F(S) \neq \phi$. If

$$\lim_{t\to\infty} ||S(t)x - S(t+k)x||$$

exists uniformly in $k > 0$, then there exists an element $p \in F(S)$ such that

$$\lim_{t\to\infty} A_t S(t+h)x = p$$

uniformly in $h \geqq 0$.

PROOF. Choose $f \in F(S)$ and set $u_t = A_t S(t)x - f$ for each $t \geqq 0$. By Lemma 3.2, put $\lim_{t\to\infty} ||u_t|| = d$. Since $\lim_{t\to\infty} ||u_{t+s} - u_t|| = 0$ for $s \geqq 0$, we have

$$\lim_{t\to\infty} ||u_{t+h} + u_t|| = 2d$$

for every $h \geqq 0$.

Now, we show that $\{A_t S(t)x\}$ is strongly convergent to an element of $F(S)$. Put

$$v(t,k) = \frac{1}{t(t+k)} \int_0^t (t-\eta)\, ||S(k+t+\eta)x - S(2(t+k)+\eta)x||\, d\eta,$$

then we have

$$A_{t+k} S(t+k)x = \frac{1}{t+k} \int_0^{t+k} A_t S(t+k+\eta)\, d\eta + v(t,k)$$

and

$$\|v(t,k)\| \leqq \frac{t}{t+k}\|x-f\|.$$

Hence,

$$\begin{aligned}
\|u_{t+k}+u_{s+k}\| &= \|A_{t+k}S(t+k)x+A_{s+k}S(s+k)x-2f\| \\
&= \left\|\frac{1}{t+k}\int_0^{t+k}\{A_tS(t+k+\eta)x+A_sS(s+k+\eta)x-2f\}d\eta\right\| \\
&\quad +\frac{t-s}{(t+k)(s+k)}\int_0^{t+k}\{A_sS(s+k+\eta)-f\}d\eta \\
&\quad +\frac{1}{s+k}\int_{t+k}^{s+k}\{A_sS(s+k+\eta)-f\}d\eta+v(t,k)+v(s,k) \\
&\leqq \frac{2}{t+k}\int_0^{t+k}\left\|\frac{1}{2}(A_tS(t+k+\eta)x+A_sS(s+k+\eta)x)-f\right\|d\eta \\
&\quad +\frac{2(s-t)}{s+k}\|x-f\|+\left(\frac{t}{t+k}+\frac{s}{s+k}\right)\|x-f\|
\end{aligned}$$

for all $s \geqq t \geqq 0$ and $k \geqq 0$. On the other hand, put

$$\begin{aligned}
\alpha_{t,s}=\sup_{h\geqq 0}\Big\| &\frac{1}{2}(A_tS(t+h)x+A_sS(s+h)x) \\
&-S(h)\left(\frac{1}{2}A_tS(t)x+\frac{1}{2}A_sS(s)x\right)\Big\|
\end{aligned}$$

then we have

$$\begin{aligned}
&\left\|\frac{1}{2}(A_tS(t+k+\eta)x+A_sS(s+k+\eta)x)-f\right\| \\
&\leqq \alpha_{t,s}+\left\|\frac{1}{2}(A_tS(t)x+A_sS(s)x)-f\right\|.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\|u_{t+k}+u_{s+k}\| \leqq 2\alpha_{t,s}+\|u_t+u_s\|+\frac{2(s-t)}{s+k}\|x-f\| \\
+\left(\frac{t}{t+k}+\frac{s}{s+k}\right)\|x-f\|
\end{aligned}$$

for all $s \geqq t \geqq 0$ and $k \geqq 0$. Letting $k \to \infty$, we have

$$2d \leqq 2\alpha_{t,s} + \|u_t + u_s\| \leqq 2\alpha_{t,s} + \|u_t\| + \|u_s\|$$

for every $t, s \geqq 0$. Since $\lim_{t,s\to\infty} \alpha_{t,s} = 0$ by Lemma 3.1, we have that $\lim_{t,s\to\infty} \|u_t + u_s\| = 2d$. By uniform convexity of X and $\lim_{t\to\infty} \|u_t\| = d$, we obtain

$$\lim_{t,s\to\infty} \|A_t S(t)x - A_s S(s)x\| = \lim_{t,s\to\infty} \|u_t - u_s\| = 0.$$

Hence $\{A_t S(t)x\}$ converges strongly. Put

$$p = \lim_{t\to\infty} A_t S(t)x.$$

Since for all $h \geqq 0$,

$$\begin{aligned} &\|A_t S(t)x - S(h) A_t S(t)x\| \\ &\leqq \frac{2\|x-f\|}{t} + \|A_t S(t+h)x - S(h) A_t S(t)x\|. \end{aligned}$$

Letting $t \to \infty$ and hence $p \in F(S)$ from Lemma 3.1. Since

$$\begin{aligned} \sup_{h\geqq 0} \|A_t S(t+h)x - p\| &\leqq \sup_{h\geqq 0} \|A_t S(t+h)x - S(h) A_t S(t)x\| \\ &\quad + \|S(h) A_t S(t)x - p\| \\ &\leqq \sup_{h\geqq 0} \|A_t S(t+h)x - S(h) A_t S(t)x\| \\ &\quad + \|A_t S(t)x - p\|. \end{aligned}$$

Letting $t \to \infty$, then $\{A_t S(t+h)x\}$ converges strongly to $p \in F(S)$ uniformly in $h \geqq 0$.

Now, we conclude this section with consequence of our previous results.

THEOREM 3.4. Let C be a closed convex subset of a uniformly convex Banach space X and $\{S(t) : t \geqq 0\}$ a nonexpansive semigroup on C. Let $x \in C$ and $F(S) \neq \phi$. If $\lim_{t\to\infty} \|S(t)x - S(t+h)x\|$ exists uniformly in $h>0$, then $\{A_t S(h)x\}$ converges strongly to a common fixed point $p \in F(S)$ uniformly in $h \geqq 0$.

PROOF. By virture of Proposition 3.3, there exists an element $p \in F(S)$ such that $\lim_{t\to\infty} A_t S(t+k)x = p$ uniformly in $k \geqq 0$. Therefore, for any $\varepsilon>0$ there exists $t_0 > 0$ such that

$$\|A_{t_0} S(t_0+k)x - p\| < \varepsilon$$

for all $k \geqq 0$. Since

$$A_t S(h)x = \frac{1}{t}\int_0^t A_{t_0} S(h+\eta)x\,d\eta + \frac{1}{tt_0}\int_0^{t_0}(t_0-\eta)\{S(\eta+h)x - S(\eta+h+t)x\}d\eta,$$

if $t \geq t_0$ then

$$\begin{aligned}\|A_t S(h)x - p\| &\leqq \frac{1}{t}\int_0^t \|A_{t_0} S(h+\eta) - p\|d\eta + \frac{t_0}{t}\|x-p\| \\ &\leqq \frac{1}{t}\int_0^{t_0} \|A_{t_0} S(h+\eta)x - p\|d\eta \\ &\quad + \frac{1}{t}\int_{t_0}^t \|A_{t_0} S(h+\eta)x - p\|d\eta + \frac{t_0}{t}\|x-p\| \\ &\leqq \frac{1}{t}\int_0^{t_0} \|A_{t_0} S(h+\eta) - p\|d\eta + \varepsilon - \frac{t_0\varepsilon}{t} \\ &\quad + \frac{t_0}{t}\|x-p\|\end{aligned}$$

for all $h \geqq 0$. Hence we have $\lim_{t\to\infty} A_t S(h)x = p$.

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