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CARTAN-THULLEN THEOREM ON $H_I(U)$

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1. Introduction

Let E be a Banach space, U be a nonempty open subset of E. $H(U)$ denotes the vector space of all complex valued holomorphic functions on U. Dineen [2] introduced b-domains of holomorphy by replacing the space $H(U)$ with the subspace $H_b(U)$ of $H(U)$ and successfully formulated a Cartan-Thullen theorem for Banach spaces.

In this article, we are going to give concepts of $H_I(U)$-domain of existence and domain of $H_I(U)$-holomorphy and prove the Cartan-Thullen theorem for infinite dimensional separable Banach space.

2. Cartan-Thullen theorem on $H_I(U)$

Let I be a countable cover of U by nonempty open subsets of U. We denote by $H_I(U)$ the vector space of all complex valued holomorphic functons on U which are bounded on each open subset of I.

The natural topology of $H_I(U)$ is the Hausdorff locally convex topology defined by the seminorms

$$p_V : f \in H_I(U) \to p_V(f) = \sup_{x \in V} |f(x)|$$

where V ranges over I.

DEFINITION 1. A nonempty connected open subset U of E is said to be a *domain of $H_I(U)$-holomorphy* if there does not exist a pair of nonempty connected open sets V and W in E such that

(a) $W \subset U \cap V$ and $V \not\subset U$.

(b) For every $f \in H_I(U)$ there exists $g \in H(V)$ such that $g|_W = f|_W$.

DEFINITION 2. Let U be a nonempty connected open subset of E. Let $f \in H_I(U)$ and $\xi \in \partial U$, where ∂U is the set of all boundary points of U. f is said to be *H_I-regular* at ξ if there exists a pair of nonempty connected open sets V, W such that $W \subset U \cap V$, $\xi \in V$ (which implies that $V \not\subset U$) and there exists $g \in H(V)$ such that $g|_W = f|_W$. Conversely, ξ is said to be a *H_I-singular points* for f if no such pair of sets exist.

f is said to be *H_I-singular* on ∂U if every point of ∂U is a H_I-singular point of f. This means that for all nonempty connected open subsets V, W of E with $W \subset U \cap V$ and $V \not\subset U$, there is no $g \in H(V)$ for which $g = f$ in W.

$S_I(U)$ will denote the set of all $f \in H_I(U)$ which are H_I-sigular at every point of ∂U. U is said to be a *$H_I(U)$-domain of existence* if $S_I(U) \neq \phi$.

LEMMA 3 [1, Proposition 8.8]. If F is a Banach space then $H_I(U; F)$ with the natural topology is a Fréchet space.

THEOREM 4. Suppose that E is a separable Banach space and let U be a nonempty connected open subset of E. Then the following are equivalent:

(a) U is a domain of $H_I(U)$-holomorphy.

(b) U is a $H_I(U)$-domain of existence.

(c) The complement $CS_I(U)$ of $S_I(U)$ in $H_I(U)$ is of first category in $H_I(U)$.

PROOF. We prove (c) implies (b) first. If $CS_I(U)$ is of first category in $H_I(U)$ then $S_I(U) \neq \phi$; for if $S_I(U)=\phi$, then $CS_I(U)=H_I(U)$ will be of first category. But by above lemma, $H_I(U)$ is a complete metric space in the natural topology, hence $H_I(U)$ is of second category. This is a contradiction.

(b) implies (a) is obvious from definitions.

Next we show that (a) implies (c). Let V and W be nonempty connected open subsets of E such that $W \subset U \cap V$ and $V \not\subset U$. $H_I(U, V, W)$ denotes the subalgebra of $H_I(U)$ consisting of all functions $f \in H_I(U)$ for which there exists a (necessarily unique) $g \in H(V)$ such that $f = g$ in W. For each $m \in N$, let $H_{I,m}(U, V, W)$ be the convex subset of $H_I(U, V, W)$ consisting of all $f \in H_I(U, V, W)$ for which the corresponding $g \in H(V)$ satisfies the relation $|g| \leqq m$ in V.

We claim that $H_{I,m}(U, V, W)$ is closed in $H_I(U)$. Since $H_I(U)$ with the natural topology is metrizable, it suffices to show that the limit of a convergent sequence in $H_{I,m}(U, V, W)$ belongs to $H_{I,m}(U, V, W)$. Let $\{f_j\}_{j\in N}$ be a sequence in $H_{I,m}(U, V, W)$, and suppose that $f_j \to f$ in $H_I(U)$ as $j \to \infty$. For each $j \in N$, let g_j be the corresponding element of $H(V)$ such that $f_j = g_j$ in W. Since $|g_j| \leqq m$ in V for every $j \in N$, it follows from Montel's theorem that $\{g_j\}_{j \in N}$ has a subsequence which converges to a $g \in H(V)$ uniformly on the compact subsets of V. In particular, $g_j \to g$ pointwise in V, and since $f_j = g_j$ in W, it follows that $f = g$ in W. Since $|g_j| \leqq m$ in V for every $j \in N$, $|g| \leqq m$ in V. Therefore $f \in H_{I,m}(U, V, W)$ and hence $H_{I,m}(U, V, W)$ is closed in $H_I(U)$ for every $m \in N$.

We claim next that the complement $CH_{I,m}(U, V, W)$ of $H_{I,m}(U, V, W)$ in $H_I(U)$ is dense $H_I(U)$, in other words, $H_{I,m}(U, V, W)$ is nowhere dense in $H_I(U)$. Since $CH_I(U, V, W) \subset CH_{I,m}(U, V, W)$, it will suffice to prove that $CH_I(U, V, W)$ is dense in $H_I(U)$. But this follows from the fact that U is a domain of $H_I(U)$-holomorphy, since then $H_I(U, V, W)$ is a proper subspace of $H_I(U)$. (The complement CG in a topological vector space H of a proper subspace G is always a dense subset of H.) Finally, we show that $CS_I(U)$ is the union of a countable family of nowhere dense sets of the form $H_{I,m}(U, V, W)$. Let M be a countable dense subset of E. If $f \in CS_I(U)$ then $f \in H_I(U, V, W)$ for some V, W. Let $g \in H(V)$ be such that $f = g$ in W, and let W_0 be the connected component of $U \cap V$ containing W. Then there exists $\xi \in V \cap \partial U \cap \partial W_0$.

Let $r > 0$ be the distance of ξ from ∂V. Then $B_r(\xi) \subset V$ and $B_r(\xi) \not\subset U$. Now choose a point $\eta \in M \cap W_0$ sufficiently close to ξ, and a rational number s sufficiently close to r so that the ball $V' = B_s(\eta)$ is contained in V, $V' \not\subset U$ and $\sup_{V'}|g| < \infty$. Let $m \in N$ be such that

$\sup_{V'}|g| \leqq m$, and let t be a sufficiently small positive rational number so that $B_t(\eta) \subset W_0$.

Let $W' = B_t(\eta)$ and $g' = g|_{V'}$. Then $f = g'$ in W', and so $f \in H_{I,m}(U, V', W')$. Since the family of sets $H_{I,m}(U, V', W')$ defined in this way is countable, $CS_I(U)$ is the union of a countable family of nowhere dense sets. Therefore $CS_I(U)$ is of first category in $H_I(U)$.

REMARK 5. It can also be shown that the Cartan-Thullen theorem holds for separable Banach spaces with the bounded approximation property [4].

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