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THE MIXED PROBLEM FOR PARABOLIC TYPE IN L^1 SPACE

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1. Introduction

Let Ω be not necessary bounded domain R^N locally regular of class C^{2m} and uniformly regular of class C^m in the sense of F. E. Browder [4]. We consider the following linear parabolic mixed problem:

(1.1) $\partial u(x,t)/\partial t + A(x,t,D)u(x,t) = f(x,t), \quad x \in \Omega, \quad 0 < t \leqq T,$

(1.2) $B_j(x,t,D)u(x,t) = 0, \quad j = 1, \cdots, m/2, \quad x \in \partial\Omega, \quad 0 < t \leqq T,$

(1.3) $u(x,0) = u_0(x), \qquad x \in \Omega$

in $L^1(\Omega)$.

Let $\{M_k\}$ be a sequence of positive numbers which has the properties specified later [5]. Assuming that $A(t)$ belongs to the class $\{M_k\}$ [5] as a function of t in some sense, we shall prove that $u(t)$ is also a function of t of the class $\{M_k\}$ provided that $f(t)$ is of the same class.

The objective of the present paper is concerned with the regularity in t of the solution $u(t)$ in $L^1(\Omega)$. Defining for each $t \in [0, T]$ a linear operator $A_p(t)$ in $L^1(\Omega)(1 < p < \infty)$ by

$$D(A_p(t))=\{u\in W'^{m,p}(\Omega):\ B_j(x,t,D)u(x)=0 \text{ on } \partial\Omega,\ j=1,\cdots,m/2\}$$

$$(A_p(t)u)(x)=A(x,t,D)u(x) \text{ for } u\in D(A_p(t)).$$

We write (1.1)-(1.3) in an evolution equation in $L^p(\Omega)$ of the following form;

$$(1.4)\qquad du(t)/dt+A_p(t)u(t)=f(t),\ 0<t\leqq T,$$

$$(1.5)\qquad u(x,0)=u_0(x).$$

In the case $1<p<\infty$ several papers have been already published concerning this problem. In [7] (see also [8], [9]) H. Tanabe proved

$$\|(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^l(\partial/\partial t)^k U(t,s)\| \leqq L_0L^{n+l+k}M_{n+l+k}(s-s)^{-n-k}$$

for some constants L_0, L and all non-negative integers n, l, and k, assuming that

i) each $A_p(t)$ is the generator of an analytic semigroup on $L^p(\Omega)$,

ii) $A_p(t)^{-1}$ is infinitely differentiable in t,

iii) the resolvents $(\lambda-A_p(t))^{-1}$ satisfy an estimation of the form

$$\|(\partial/\partial t)(\lambda-A_p(t))^{-1}\|\leqq K_0K^nM_n/|\lambda|$$

for non-negative integer n and $\lambda\in\Sigma=\{\lambda:\ \theta\leqq\arg\lambda\leqq 2\pi-\theta,\ 0<\theta<\pi/2\}$.

In the case of $p=1$, in the paper [6] we have already constructed the evolution operator $U(t,s)$ of the above equation.

The main theorem of this paper is to establish a similar esitmation in $L^1(\Omega)$.

We start in Section $\cdot$2 with some preliminaries and assumptions related to the paper. In Section 3, we study the case $m > N/2$. Finally, in Section 4, we simply establish the estimate of the main result.

2. Preliminaries and assumptions

If X and Y are the Banach spaces, we denote by $B(X, Y)$ the set of all bounded linear operators from X to Y. The operator norm of $A(t)$ is denoted by $A(t)_{B(X,Y)}$.

Let $W^{m,p}(\Omega)$ be the Banach space consisting of measurable functions defined in Ω whose distribution derivatives of order up to m belong to $L^p(\Omega)$. The norm of $W^{m,p}(\Omega)$ and $L^p(\Omega)$ is defined by

$$u_{m,p} = \Big(\sum_{|\alpha| \leqq m} \int_\Omega |D^\alpha u| dx \Big)^{1/p}$$

$$u_p = \Big(\int |u|^p dx \Big)^{1/p}$$

respectively.

For each $t \in (0, T]$,

$$A(x, t, D) = \sum_{|\alpha| \leqq m} a_\alpha(x, t) D^\alpha$$

is a strongly elliptic linear differential operator of order m and

$$B_j(x, t, D) = \sum_{|\beta| \leqq m_j} b_{j,\beta}(x, t) D^\beta, \quad j = 1, \cdots, m/2$$

is a normal set of linear differential operators on $\partial\Omega$ of an order less than m.

Let $A(t)$ be an operator in $L^p(\Omega)$, $1 < p < \infty$, defined by

(2.1) $D(A(t))=\{u\in W^{m,p}(\Omega): B_j(x,t,D)u(x)=0,$
$j=1,\cdots,m/2,\ x\in\partial\Omega\},$
for $x\in D(A(t))$, $(A(t)u)(x)=A(x,t,D)u(x)$.

REMARK. Properly speaking, $A(t)$ depends on p and we would normally write $A_p(t)$ but here, for simplicity we will write $A(t)$.

Let $\arg\lambda=\theta$ be called the *ray of minimal growth* of the resolvent of $A(t)$ in the sense of S. Agmon for any $\phi\in(\pi/2,\ 3\pi/2)$[9].

(A.1) If $B_j(x,t,D)_{j=1}^{m/2}$ is of Dirichlet type then $-A(t)$ for any strongly elliptic operator $A(x,t,D)$ of order m generates an analytic semigroup $\exp(-\tau A(t))$ in $L^p(\Omega)$:
i.e., there exists $\theta_0\in(0,\pi/2)$ such that

$$\rho(A(t))\supset\Sigma=\{\lambda:\theta_0\leqq\arg\lambda\leqq2\pi-\theta_0,|\lambda|\geqq C_0\}$$

then $C_0=0$ implies $0\in\rho(A(t))$.

(A.2) The formal adjoint of $A(x,t,D)$ is

$$A'(x,t,D)=\sum_{|\alpha|\leqq m}a_\alpha'(x,t)D^\alpha,$$

and the adjoint system of boundary operators $\{B_j(x,t,D)\}_{j=1}^{m/2}$ can be constructed for $A(x,t,D)$ of $\{B_j(x,t,D)\}_{j=1}^{m/2}$.

REMARK. If $A'(t)$ defines $\{A'(x,t,D),\ B_j'(x,t,D),p'\}$ replacing $\{A(x,t,D),\ B_j(x,t,D),p\}$ in (2.1) then $A^*(t)=A'(t)$, where $A^*(t)$ is the adjoint system of $A(t)$ defined in $L^p(\Omega)$.

Let $\{M_k: k=0,1,2,\cdots\}$ be a sequence of positive numbers satisfying the following condition [5]: there exist positive numbers d_0, d_1, and d_2 such that

$$M_{k+1} \leqq d_0^k M_k, \quad M_k \leqq M_{k+1}, \ k = 0, 1, 2, \cdots,$$
$$\binom{k}{j} M_{k-j} M_j \leqq d_1 M_k, \quad 0 \leqq j \leqq k,$$
$$M_{j+k} \leqq d_2^{j+1} M_j M_k, \quad j, k = 0, 1, 2, \cdots.$$

Here, however, all of the coefficients of $A(x,t,D)$ and $\{B_j(x,t,D)\}_{j=1}^{m/2}$, as functions of t, are assumed to belong to the class $\{M_k\}$.

(A. 3) The inequalities

$$|(\partial/\partial t)^l a_\alpha(x,t)| \leqq B_0 B^l M_l,$$
$$|(\partial/\partial t)^l D_x^\gamma \ b_{j,\beta}(x,t)| \leqq B_0 B^l M_l$$

hold for every $x \in \bar{\Omega}$, $t \in (0, T]$, $|\alpha| \leqq m$, $|\beta| \leqq m_j$, $|\gamma| \leqq m - m_j$, $j = 1, \cdots, m/2$ and $l = 0, 1, 2, \cdots$, there exist positive numbers B_0 and B.

Under the above assumption we consider the following: there exist positive numbers K_0 and K such that for every $\lambda \in \Sigma$, and $l = 0, 1, 2, \cdots$

$$\|(\partial/\partial t)^l (A(t) - \lambda)^{-1}\|_{B(L^1, L^1)} \leqq K_0 K^l M_l.$$

Putting

$$A(x,t,D+\eta) = \sum_{|\alpha| \leqq m} a_\alpha(x,t)(D+\eta)^\alpha,$$
$$B_j(x,t,D+\eta) = \sum_{|\beta| \leqq m_j} b_{j,\beta}(x,t)(D+\eta)^\beta \quad \text{for } \eta \in \boldsymbol{C}^N.$$

The operators $A^\eta(t)$and $A'^\eta(t)$ are defined as follows:

$$D(A^\eta(t)) = \{u \in W^{m,p}(\Omega) : B_j(x,t,D+\eta)u(x) = 0,$$
$$j = 1, \cdots, m/2, \ x \in \partial\Omega\}$$
$$(A^\eta(t)u)\ (x) = A(x,t,D+\eta)u(x) \ \text{ for } u \in D(A^\eta(t)),$$

and

$$D(A'^{\bar\eta}(t))=\{u\in W^{m,p'}(\Omega):\ B_j'(x,t,D+\bar\eta)u(x)=0,$$
$$j=1,\cdots,\ m/2,\ \ x\in\partial\Omega\}$$
$$(A'^{\bar\eta}(t)u)(x)=A'(x,t,D+\bar\eta)u(x)\ \text{ for } u\in D(A'^{\bar\eta}(t)).$$

Then

$$(A^\eta(t))^*=A'^{\bar\eta}(t)[9].$$

If $\delta>0$ is a sufficiently small, $\lambda\in\Sigma$ and $|\eta|\leqq\delta|\lambda|^{1/m}$ then $\lambda\in\rho(A^\eta(t))$ we get [9]

$$||(A^\eta(t)-\lambda)^{-1}||_{B(L^p,L^p)}\leqq C_p/|\lambda|,$$
$$||(A^\eta(t)-\lambda)^{-1}||_{B(L^p,W^{m,p})}\leqq C_p,$$
$$||(A'^{\bar\eta}(t)-\lambda)^{-1}||_{B(L^{p'},W^{m,p'})}\leqq C_p.$$

PROPOSITION 2.1. There exist positive numbers C_1 and C_2 such that for every $\lambda\in\Sigma$ and $l=0,1,2,\cdots$

$$||(\partial/\partial t)^l(A^\eta(t)-\lambda)^{-1}||_{B(L^p,L^p)}\leqq C_1C_2^lM_l/|\lambda|,$$
$$||(\partial/\partial t)^l(A^\eta(t)-\lambda)^{-1}||_{B(L^p,W^{m,p})}\leqq C_1C_2^l\ M_l,$$
$$||(\partial/\partial t)^l(A'^{\bar\eta}(t)-\lambda)^{-1}||_{B(L^{p'},W^{m,p'})}\leqq C_1C_2^lM_l.$$

PROOF. If there is no fear of confusion we simply write $||\cdot||_m$ and $||\cdot||$ in place of $||\cdot||_{m,p}$ and $||\cdot||_p$ respectively. For $f\in L^p(\Omega)$, putting

$$u(t)=(A^\eta(t)-\lambda)^{-1}f$$

we get

$$(2.2)\quad (A(x,t,D+\eta)-\lambda)u(x,t)=f(t),\ \ x\in\Omega,$$
$$(2.3)\quad B_j(x,t,D+\eta)u(x,t)=0,\ \ x\in\partial\Omega,\ \ j=1,\cdots,m/2.$$

Putting

$$u^l=(\partial/\partial t)^lu,$$
$$A^l(x,t,D+\eta)=\sum_{|\alpha|\leqq m}(\partial/\partial t)^la_\alpha(x,t)(D+\eta)^\alpha,$$

$$B_j^l(x,t,D+\eta) = \sum_{|\beta|\leqq j} (\partial/\partial t)^l b_{j,\beta}(x,t)(D+\eta)^l,$$
$$j=1, \cdots, m/2.$$

And differentiatng in t both sides of (2.2) and (2.3) we get

$$(A(x,t,D+\eta)-\lambda)u^l(x,t) = -\sum_{k=0}^{l-1}\binom{l}{k}A^{l-k}(x,t,D+\eta)u^k(x,t),$$

$$B_j(x,t,D+\eta)u^l(x,t) = -\sum_{k=0}^{l-1}\binom{l}{k}B_j^{l-k}(x,t,D+\eta)u^k(x,t).$$

In view of these we get

$$\begin{aligned}
&(A(x,t,D)-\lambda)u^l(x,t) \\
&= (A(x,t,D)-A(x,t,D+\eta))u^l(x,t) \\
&\quad -\sum_{k=0}^{l-1}\binom{l}{k}A^{l-k}(x,t,D+\eta)u^k(x,t), \\
&B_j(x,t,D)u^l(x,t) \\
&= (B_j(x,t,D)-B_j(x,t,D+\eta))u^l(x,t) \\
&\quad -\sum_{k=0}^{l-1}\binom{l}{k}B_j^{l-k}(x,t,D+\eta)u^k(x,t).
\end{aligned}$$

By Theorem 17.5 of [9], we get the inequality

$$\begin{aligned}
&\sum_{i=0}^{m} |\lambda|^{(m-i)/m}\|u^l(t)\|_i \\
&\leqq C\Big\{\Big\|(A(x,t,D)-A(x,t,D+\eta))u^l(t) \\
&\quad -\sum_{k=0}^{l-1}\binom{l}{k}A^{l-k}(x,t,D+\eta)u^k(t)\Big\| \\
&\quad +\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\Big\|(B_j(x,t,D)-B_j(x,t,D+\eta))u^l(t) \\
&\quad -\sum_{k=0}^{l-1}\binom{l}{k}B_j^{l-k}(x,t,D+\eta)u^k(t)\Big\| \\
&\quad +\sum_{j=1}^{m/2}\Big\|(B_j(x,t,D)-B_j(x,t,D+\eta))u^l(t) \\
&\quad -\sum_{k=0}^{l-1}\binom{l}{k}B_j^{l-k}(x,t,D+\eta)u^k(t)\Big\|_{m-m_j}\Big\}.
\end{aligned}$$

Putting $\delta \leqq 1$, we get

$$||(A(x,t,D)-A(x,t,D+\eta))u^l(t)||$$

$$\leqq C\sum_{i=1}^{m-1}|\eta|^{m-i}||u(t)||_i \leqq C\sum_{i=0}^{m-1}(\delta|\lambda|^{1/m})^{m-i}||u(t)||_i$$

$$\leqq C\sum_{i=0}^{m-1}|\lambda|^{(m-i)/m}||u(t)||_i\,,$$

$$\sum_{k=0}^{l-1}\binom{l}{k}A^{l-k}(x,t,D+\eta)u^k(t)$$

$$\leqq C\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m}|\eta|^{m-i}||u^k(t)||_i$$

$$\leqq C\sum_{k=0}^{l-k}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m}|\lambda|^{(m-k)/m}||u^k(t)||_i,$$

$$\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}||(B_j(x,t,D)-B_j(x,t,D+\eta))u^l(t)||$$

$$\leqq C\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\sum_{i=0}^{m_j-1}|\eta|^{m_j-i}||u^l(t)||_i$$

$$\leqq C\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\sum_{i=0}^{m_j-1}(\delta|\lambda|^{1/m})^{m_j-i}\,||u^l(t)||_i$$

$$\leqq C\delta\sum_{i=0}^{m-1}|\lambda|^{(m-i)/m}\,||u^l(t)||_i,$$

$$\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\left|\left|\sum_{k=0}^{l-1}\binom{l}{k}\,B_j{}^{l-k}(x,t,D+\eta)u^k(t)\right|\right|$$

$$\leqq C\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m_j}|\eta|^{m_j-i}||u^k(t)||_i$$

$$\leqq C\sum_{j=1}^{m/2}|\lambda|^{(m-m_j)/m}\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m_j}|\lambda|^{(m_j-i)/m}||u^k(t)||_i$$

$$\leqq C\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=1}^{m}|\lambda|^{(m-i)/m}||u^k(t)||_i,$$

$$\sum_{j=1}^{m/2} \|(B_j(x,t,D)-B_j(x,t,D+\eta))u^l(t)\|_{m-m_j}$$

$$\leqq C\sum_{j=1}^{m/2}\sum_{i=0}^{m_j-1}|\eta|^{m_j-i}\|u^l(t)\|_{m-m_j+i}$$

$$\leqq C\sum_{i=0}^{m-1}|\eta|^{m-i}\|u^l(t)\|_i$$

$$\leqq C\delta\sum_{i=0}^{m-1}|\lambda|^{(m-i)/m}\|u^l(t)\|_i$$

$$\sum_{j=1}^{m/2}\left\|\sum_{k=0}^{l-1}\binom{l}{k}B_j^{l-k}(x,t,D+\eta)u^k(t)\right\|_{m-m_j}$$

$$\leqq C\sum_{j=1}^{m/2}\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m_j}|\eta|^{m_j-i}\|u^k(t)\|_{m-m_j+i}$$

$$\leqq C\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|u^k(t)\|_i.$$

Inserting these into (2.4) we get

$$\sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|u^l(t)\|_i$$

$$\leqq C\Big\{\delta\sum_{i=0}^{m-1}|\lambda|^{(m-i)/m}\|u^l(t)\|_i$$

$$+\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|u^k(t)\|_i\Big\}.$$

Replacing $\delta>0$ by sufficiently small number if necessary we get

$$\text{(2.5)}\qquad \sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|u^l(t)\|_i$$

$$\leqq C\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}\sum_{i=0}^{m}|\lambda|^{(m-i)/m}\|u^k(t)\|_i.$$

Applying $\|u(t)\|_i\leqq C\|u(t)\|_m^{i/m}\,\|u(t)\|^{(m-i)/m}$ and Young's inequality to the above inequality and

$$(2.6)\quad \|u^l(t)\|_m+|\lambda|\ \|u^l(t)\|$$
$$\leqq C\sum_{k=0}^{l-1}\binom{l}{k}B_0B^{l-k}M_{l-k}(\|u^k(t)\|_m+|\lambda|\ \|u^k(t)\|).$$

Hence, (2.5) and (2.6) are essentially equivalent.

In view of (7) there exist positive numbers C_1 and C_2 such that

$$\|u^l(t)\|_m+|\lambda|\ \|u^l(t)\|\leqq C_1C_2M_l\|f\|.$$

By the above inequality, we obtain the conclusion of proposition.

3. Estimates of the kernel of the derivatives of $\exp(-\tau A(t))$

For simplicity we consider $m>N/2$ and by Sobolev's imbedding theorem we get

$$(3.1)\quad \|u(t)\|_\infty\leqq\gamma\,\|u(t)\|_{m,2}^{N/2m}\,\|u(t)\|_2^{1-N/2m}.$$

We consider that $K_{\lambda,\mu}(x,y:t)$and $K^\eta_{\lambda,\mu}(x,y;t)$ denote the kernel of $(A(t)-\lambda)^{-1}(A(t)-\mu)^{-1}$ and $(A^\eta(t)-\lambda)^{-1}(A^\eta(t)-\mu)^{-1}$ respectively for $\lambda,\mu\in\Sigma$ and $|\eta|\leqq\delta\ \min(|\lambda|^{1/m},|\mu|^{1/m})$.

LEMMA 3.1. There exist positive numbers C_1 and C_2 such that for any $\lambda,\mu\in\Sigma$ and $l=0,1,2\cdots$

$$|(\partial/\partial t)^l K^\eta_{\lambda,\mu}(x,y:t)|$$
$$\leqq\gamma^2C_1{}^2C_2{}^ld_1(l+1)M_l|\lambda|^{N/2m-1}|\mu|^{N/2m-1}.$$

PROOF. In view of Leibniz's formula we get

$$(\partial/\partial t)^l\{(A^\eta(t)-\lambda)^{-1}(A^\eta(t)-\mu)^{-1}\}$$
$$=\sum_{k=0}^{l} lk(\partial/\partial t)^{l-k}(A^\eta(t)-\lambda)^{-1}(\partial/\partial t)^k(A^\eta(t)-\mu)^{-1}.$$

In view of (3. 1) and Proposition 2. 1 we get

$$\|(\partial/\partial)^{l-k}(A^\eta(t)-\lambda)^{-1}f\|_\infty$$
$$\leqq\gamma\|(\partial/\partial t)^{l-k}(A^\eta(t)-\lambda)^{-1}f\|_m^{N/2m}$$
$$\|(\partial/\partial t)^{l-k}(A^\eta(t)-\lambda)^{-1}f\|^{1-N/2m}$$
$$\leqq\gamma(C_1C_2{}^{l-k}M_{l-k}\|f\|)^{N/2m}(C_1C_2{}^{l-k}M_{l-k}\|f\|/|\lambda|)^{1-N/2m}$$
$$=\gamma C_1C_2{}^{l-k}M_{l-k}|\lambda|^{N/2m-1}\|f\|.$$

Hence,

$$\|(\partial/\partial t)^{l-k}(A^\eta(t)-\lambda)^{-1}\|_{B(L^2,L^\infty)}\leqq\gamma C_1C_2{}^{l-k}M_{l-k}|\lambda|^{N/2m-1}.$$

In view of (9) we get

$$|\text{Kernel of }(\partial/\partial t)^{l-k}(A^\eta(t)-\lambda)^{-1}(\partial/\partial t)^k(A^\eta(t)-\mu)^{-1}|$$
$$\leqq\|(\partial/\partial)^{l-k}(A^\eta(t)-\lambda)^{-1}\|_{B(L^2,L^\infty)}$$
$$\|(\partial/\partial t)^k(A'^\eta(t)-\mu)^{-1}\|_{B(L^2,L^\infty)}$$
$$\leqq\gamma C_1C_2{}^{l-k}M_{l-k}|\lambda|^{N/2m-1}C_1C_2{}^kM_k|\mu|^{N/2m-1}$$
$$=\gamma^2C_1{}^2C_2{}^lM_{l-k}M_k|\lambda|^{N/2m-1}|\mu|^{N/2m-1}.$$

Thus

$$|(\partial/\partial t)^lK^\eta_{\lambda,\mu}(x,y:t)|$$
$$\leqq\sum_{k=0}^{l}\binom{l}{k}\gamma^2C_1C_2{}^lM_{l-k}M_k|\lambda|^{N/2m-1}|\mu|^{N/2m-1}$$
$$\leqq\gamma^2C_1{}^2C_2{}^ld(l+1)M_l|\lambda|^{N/2m-1}|\mu|^{N/2m-1}.$$

Therefore the proof of Lemma 3. 1 is completed.

LEMMA 3. 2. There exist positive numbers C_1 and C_2 such that for any $\lambda,\mu\in\Sigma$ and $l=0,1,2,\cdots$

$$|(\partial/\partial t)^lK_{\lambda,\mu}(x,y:t)|$$

$$\leqq \gamma^2 C_1^2 C_2^l d_1(l+1) M_l |\lambda|^{N/2m-1} |\mu|^{N/2m-1} \{ e^{-\delta|\lambda|^{1/m}|x-y|} + e^{-\delta|\mu|^{1/m}|x-y|} \}.$$

PROOF. $K_{\lambda,\mu}(x, y:t) = e^{-(x-y)\eta} K^{\eta}_{\lambda,\mu}(x, y:t)$. (1)
Hence, for real vector η, we get

$$|(\partial/\partial t)^l K_{\lambda,\mu}(x, y:t)| = |e^{-(x-y)\eta} (\partial/\partial t)^l K^{\eta}_{\lambda,\mu}(x, y:t)| \leqq e^{-(x-y)\eta} \gamma^2 C_1 C_2^l d_1(l+1) M_l |\lambda|^{N/2m-1} |\mu|^{N/2m-1}.$$

Inserting the minimal value of η into the right side of the inequality we get

$$|(\partial/\partial t)^l K_{\lambda,\mu}(x, y:t)| \leqq \gamma^2 C_1 C_2^l d_1(l+1) M_l |\lambda|^{N/2m-1} |\mu|^{N/2m-1} \; e^{-\delta \min(|\lambda|^{1/m},\, |\mu|^{1/m})|x-y|}.$$

In view of this and

$$e^{-\delta \min(|\lambda|^{1/m}, |\mu|^{1/m})|x-y|} \leqq e^{-\delta|\lambda|^{1/m}|x-y|} + e^{-\delta|\mu|^{1/m}|x-y|}.$$

PROPOSITION 3.3. There exist positive numbers C_3, C_4, c and $\theta \in (0, \pi/2)$ such that for any $|\arg\tau| \leqq \theta_1$ and $l = 0, 1, 2, \cdots$

$$|(\partial/\partial t)^l G(x, y, \tau:t)| \leqq C_3 C_4^l (M_l \frac{1}{|\tau|^{N/m}} \exp\left(-c \frac{x-y^{m/(m-1)}}{|\tau|}\right).$$

PROOF. We get

$$\exp(-\tau A(t)) = \exp(-(\tau/2) A(t))^2$$
$$= \frac{1}{2\pi i} \int_{\Gamma} e^{-(\tau/2)\lambda} (A(t)-\lambda)^{-1} d\lambda \frac{1}{2\pi i} \int_{\Gamma} e^{-(\tau/2)\mu} (A(t)-\mu)^{-1} d\mu$$
$$= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma} \int_{\Gamma} e^{-(\tau/2)(\lambda+\mu)} (A(t)-\lambda)^{-1} (A(t)-\mu)^{-1} d\lambda d\mu.$$

Hence

$$G(x, y, \tau:t) = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma} \int_{\Gamma} e^{-(\tau/2)(\lambda+\mu)} K_{\lambda,\mu}(x, y:t) d\lambda d\mu$$

where $\Gamma_{x,y,\tau/2}=\{\lambda:|\arg\lambda|=\theta_2,|\lambda|\geqq a\}=\{\lambda:\lambda=ae^{i\theta},|\theta|\geqq\theta_0\}$.

$$a=\varepsilon\frac{|x-y|^{m/(m-1)}}{|\tau/2|^{m/(m-1)}},\quad \rho=\frac{|x-y|^{m/(m-1)}}{|\tau/2|^{1/(m-1)}},\quad a=\frac{\varepsilon\rho}{|\tau/2|}$$

and $a^{1/m}|x-y|=\varepsilon^{1/m}\rho$.

In view of Lemma 3.2 we get

(3.2) $\quad |(\partial/\partial t)^l G(x,y,\tau:t)|$

$$\leqq\left|\left(\frac{1}{2\pi i}\right)^2\right|\int_{\Gamma_{x,y,\tau/2}}\int_{\Gamma_{x,y,\tau/2}}|e^{-(\tau/2)(\lambda+\mu)}||(\partial/\partial t)^l K_{\lambda,\mu}(x,y:t)||$$

$$||d\lambda||d\mu|$$

$$=\left(\frac{1}{2\pi}\right)^2\gamma^2C_1^2C_2^l\left\{\int_{\Gamma_{x,y,\tau/2}}|\lambda|^{N/2m-1}e^{-Re(\tau/2)\lambda}e^{-\delta|\lambda|^{1/m}|x-y|}|d\lambda|\times\right.$$

$$\int_{\Gamma_{x,y,\tau/2}}|\mu|^{N/2m-1}e^{-Re(\tau/2)\mu}|d\mu|$$

$$=\int_{\Gamma_{x,y,\tau/2}}|\lambda|^{N/2m-1}e^{-Re(\tau/2)\lambda}|d\lambda|\times$$

$$\left.\int_{\Gamma_{x,y,\tau/2}}|\mu|^{N/2m-1}e^{-Re(\tau/2)}e^{-\delta|\mu|^{1/m}|x-y|}|d\mu|\right\}.$$

For $0<\theta_0<1$, if τ is such that $\dfrac{|Im\tau|}{Re\tau}\leqq(1-\theta_0)\dfrac{\cos\theta_0}{\sin\theta_0}$ assuming $\lambda=re^{\pm i\theta_0}(r>0)$ then $Re(\tau/2)^\lambda\leqq rRe(\tau/2)\varepsilon_0\cos\theta_0$. Hence, there exists a positive c such that

$$Re(\tau/2)\lambda\geqq cr|\tau/2|\ [9],$$

we take $\Gamma_{x,y,\tau/2}=\Gamma_1\Gamma_2\Gamma_3$, where $\Gamma_1=\{\lambda=re^{-i\theta_0}:r\geqq a\}$, $\Gamma_2=\{\lambda=ae^{i}:\theta_0\leqq\phi\leqq2\pi-\theta_0\}$ and $\Gamma_3=\{\lambda=re^{i\theta_0}:r\geqq a\}$.

We get

$$\int_{\Gamma_3}|\lambda|^{N/2m-1}e^{-Re(\tau\lambda/2)}e^{-\delta|\lambda|^{1/m}|x-y|}|d\lambda|$$

$$\leqq\int_a^\infty r^{N/2m-1}e^{-cr|\tau/2|}e^{-\delta r^{1/m}|x-y|}dr$$

$$\leqq e^{-ca^{1/m}|x-y|}\int_a^\infty r^{N/2m-1}e^{-cr|\tau/2|}dr$$
$$\leqq \Gamma(N/2m)((1/c)|2/\tau|)^{N/2m}e^{-\delta\varepsilon^{1/m}}\rho.$$

And similarly for an integral along Γ_1,

$$\int_{\Gamma_2}|\lambda|^{N/2m-1}e^{-Re(\tau\lambda/2)}e^{-\delta|\lambda|^{1/m}|x-y|}|d\lambda|$$
$$\leqq a^{N/2m-1}e^{|\tau/2|a}e^{-\delta a^{1/m}|x-y|}2\pi a$$
$$=2\pi a^{N/2m}e^{|\tau/2|a}e^{-\delta a^{1/m}|x-y|}.$$

If x and y are positive numbers, then $x^y\leqq(y/e)^y e^x$. Hence, the integral of the same function along Γ_2 is dominated by

$$2\pi|2/\tau|^{N/2m}(N/2me)^{N/2m}e^{2\varepsilon\delta}e^{-\delta\varepsilon^{1/m}}\rho.$$

Collecting these results we obtain

$$\int_{\Gamma x,y,\tau/2}|\lambda|^{N/2m-1}e^{Re(\tau\lambda/2)}e^{-\delta|\lambda|^{1/m}|x-y|}|d\lambda|$$
$$\leqq 2\Gamma(N/2m)((1/c)|2/\tau|^{N/2m}e^{-\delta\varepsilon^{1/m}}\rho+$$
$$2\pi|2/\tau|^{N/2m}(N/2m\rho)^{N\ 2m}e^{2\varepsilon\rho-\delta\varepsilon^{1/m}\rho}$$
$$\leqq\{2c^{-N/2m}\Gamma(N/2m)+2\pi(N/2me)^{N/2m}\}|2/\tau|^{N/2m}e^{2\varepsilon\rho\delta\varepsilon^{1/m}\rho}.$$
$$\int_{\Gamma_3}|\mu|^{N/2m}e^{-Re(\tau\mu/2)}|d\mu|\leqq\Gamma(N/2m)((1/c)|2/\tau|)^{N/2m}.$$
$$\int_{\Gamma_2}|\mu|^{N/2m-1}e^{-Re(\tau\mu/2)}|d\mu|\leqq 2\pi|2/\tau|^{N/2m}(N/2me)^{N/2m}e^{2\varepsilon\rho}.$$

Inserting these into (3.2) we get

$$|(\partial/\partial t)^l G(x,y,\tau:t)|$$
$$\leqq 2(1/2\pi)^2\gamma^2 C_1{}^2 C_2{}^l d_1(l+1)M_l\{2c^{-N/2m}\Gamma(N/2m)+$$
$$2\pi(N/2me)^{N/2m}\}^2|2/\tau|^{N\ m}\times e^{4\varepsilon\rho-\delta\varepsilon^{1/m}\rho}.$$

If $\varepsilon>0$ is sufficiently small then $\delta\varepsilon^{1/m}-4\varepsilon>0$, and

$$4\varepsilon\rho-\delta\varepsilon^{1/m}\rho=-2^{1/(m-1)}(\delta\varepsilon^{1/m}-4\varepsilon)\frac{|x-y|^{m/(m-1)}}{|\tau|^{1/(m-1)}}.$$

Therefore, $l+1<e^l$ holds and we obtain the conclusion of proposition.

4. The main theorem

Now, denote the kernel of $(A(t)-\lambda)^{-1}$ by $K_\lambda(x,y:t)$

PROPOSTION 4.1. There exist positive numbers C_5, C_6 and $\theta_0 \in (0,\pi/2)$ such that for any $\arg\lambda \in (-\theta_0, \theta_0)$ and $l=0,1,2,\cdots$

$$|(\partial/\partial t)^l K_\lambda(x,y:t)|$$

$$\leqq C_5 C_6^l M_l e^{-\delta|\lambda|^{1/m}|x-y|} \times \begin{cases} |x-y|^{m-N} & \text{if } m<N \\ |\lambda|^{N/m-1} & \text{if } m>N \\ 1+\log^+(|\lambda|^{1/m}|x-y|)^{-1} & \\ & \text{if } m=N. \end{cases}$$

PROOF. The proof of this proposition is similar to [9]. We write (1.1)-(1.3) as an evolution equation in $L^1(\Omega)$:

(4.1) $du(t)/dt + A(t)u(t) = f(t), \quad 0<t\leqq T,$

(4.2) $u(0) = u_0.$

Let $U(t,s)$ be the evolution operator of (4.1) which is a bounded operator valued function defined in $\bar{\Delta}$ satisfying

$$\begin{aligned} &\partial U(t,s)/\partial t + A(t)U(t,s) = 0 && \\ &\partial U(t,s)/\partial s + U(t,s)A(s) = 0 && (ts)\in\Delta \\ &U(s,s) = I && 0\leqq s\leqq T, \end{aligned}$$

where $\Delta = \{(s,t): 0\leqq s<t\leqq T$ and $\bar{\Delta} = \{(s,t): 0\leqq s\leqq t\leqq T\}$. The existence of such an operator is known by [6].

THEOREM. Under the assumptions stated above the evoluion operator $U(t,s)$ of (4.1) is infinitely differentiable in $(s,t)\in\Delta$. There exist constants L_0, L such that

$$\|(\partial/\partial t)^n(\partial/\partial t+\partial/\partial s)^m(\partial/\partial s)^k\|_{B(L^1,L^1)}$$
$$\leqq L_0L^{n+m+k}M_{n+m+k}(t-s)^{-n-k}, \qquad (s,t)\in\Delta$$

for $n, m, k=0, 1, 2, \cdots$.

According to [7] it suffices to prove the following Proposition 4.2. in order to establish the above Theorem.

PROPOSITION 4.2. There exist positive numbers K_0 and K such that

$$\|(\partial/\partial t)^l(A(t)-\lambda)^{-1}\|_{B(L^1,L^1)}\leqq K_0K^lM_l/|\lambda|$$

for $l=0, 1, 2, \cdots$.

PROOF. In view of Proposition 4.1. we obtain

$$\int_\Omega\int_\Omega|(\partial/\partial t)^lK_\lambda(x,y:t)f(y)dy|dx$$
$$\leqq C_5C_6^lM_l\int_\Omega\int_\Omega e^{-\delta|\lambda|^{1/m}|x-y|}|x-y|^{m-N}|f(y)|dydx$$
$$\leqq C_5C_6^lM_l\int_\Omega\int_{R^N} e^{-\delta|\lambda|^{1/m}|x-y|}|x-y|^{m-N}dx|f(y)|dy$$
$$\leqq C_5C_6^lM_l\int_\Omega\int_0^\infty e^{-\delta|\lambda|^{1/m}|r|}r^{m-N}r^{N-1}dr|f(y)|dy$$
$$=C_5C_6^lM_l\int_\Omega\int_0^\infty e^{-\delta\rho}\rho^{m-1}d\rho|\lambda|^{-1}|f(y)|dy$$
$$=K_0K^lM_l\|f\|_{L^1}/|\lambda|$$

where $K_0=C_5e^{-\delta\rho}$ and $K=C_6$.

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