PUSAN KYŎNGNAM MATHEMATICAL JOURNAL
Vol. 4, 193~201, 1988

MINMAX THEOREMS AND LOCAL OPTIMIZATION PROBLEMS

JONG YEOUL PARK

In this paper, we prove the generalized minmax theorem and by using of topological degree we prove the local dotimization problem.

LEMMA 1. Let X be a compact subset of a topological vector space E, Y be a convex subset of a topological vectors space E and let F be a real valued function on $X\times Y$ satisfying:

(1) For each $y\in Y$, the function $F(x,y)$ of x is upper semicontinuous;

(2) for each $x\in X$, the function $F(x,y)$ of y is convex;

(3) for any constant c, $\sup_{x\in X}\inf_{y\in Y} F(x,y)<c$.

Then there exists a continuous mapping p of X into Y such that $F(x,p(x))<c$ for all $x\in X$.

PROOF. By (3), for every $x\in X$ there exists $y_c\in Y$ such that $F(x,y_c)<c$. Setting

$$A_{y_c}=\{x\in X: F(x,y_c)<c\}$$

for each $y_c\in Y$, thus we have $X=\bigcup_{y_c\in Y} A_{y_c}$. Since X is compact, there exists a finite family $\{y_1,y_2,\cdots y_n\}$ such that

$X = \sum_{i=1}^{n} A_{y_i}$. Let $\{\beta_1, \beta_2, \cdots, \beta_n\}$ be a partion of unity corresponding to this convering, i.e., each β_i is a continuous mapping of X into $[0, 1]$ which vanishes outside of A_{y_i}, while $\sum_{i=1}^{n} \beta_i = 1$ for all $x \in X$. For each i such that $\beta_i(x) = 0$, x lies in A_{y_i}, so that $F(x, y_i) < c$. Hence, we have

$$\sum_{i=1}^{n} \beta_i(x) F(x, y_i) < c$$

for all $x \in X$. Define a continuous mapping p of X into Y by setting

$$p(x) = \sum_{i=1}^{n} \beta_i(x) y_i.$$

By convexity, we see that

$$F(x,\ p(x)) = F\Big(x,\ \sum_{i=1}^{n} \beta_i(x) y_i\Big)$$

$$\leqq \sum_{i=1}^{n} \beta_i(x) F(x, y_i) < c$$

for all $x \in X$. Thus, there exists a continuous mapping p of X into Y such that $F(x, p(x)) < c$ for all $x \in X$.

LEMMA 2. Let Y be a compact subset of a topological vector space E, X be a convex subset of a topological vector space E and let F be a real valued function on $X \times Y$ satisfying:

(1) For each $y \in Y$ the function $F(x, y)$ of x is concave;
(2) for each $x \in X$, the function $F(x, y)$ of y is lower

semicontinuous;

(3) for any constant c, $\inf_{y \in Y} \sup_{x \in X} F(x, y) > c$.

Then, there exists a continuous mapping q of Y into X such that $F(q(y), y) > c$ for all $y \in Y$.

PROOF. Since Y is compact, there exists a finite family $\{x_1, x_2, \cdots, x_n\}$ such that $Y = \bigcap_{i=1}^{n} C_{x_i}$. Let $\{r_1, r_2, \cdots, r_n\}$ be a partition of unity corresponding to this covering, i.e., each r_i is a continuous mapping of X into $[0, 1]$ which vanishes outside of C_{x_i}, while $\sum_{i=1}^{n} r_i = 1$ for all $y \in Y$. For each i, such that $r_i(y) = 0$, y lies in C_{x_i}. So ~~that~~ $F(x_i, y) > c$. Hence, we have

$$\sum_{i=1}^{n} r_i(y) F(x_i, y) > c$$

for all $y \in Y$.

Define a continuous mapping q of Y into X by setting

$$q(y) = \sum_{i=1}^{n} r_i(y) x_i.$$

By concavity, we see that

$$F(q(y), y) = F\Big(\sum_{i=1}^{n} r_i(y) x_i, y\Big)$$
$$\geqq \sum_{i=1}^{n} r_i(y) F(x_i, y) > c$$

for all $y \in Y$. Thus, there exists a continuous mapping q of Y into X such that $F(q(y), y) > c$ for all $y \in Y$.

THEOREM 1. Let X and Y be compact convex subsets each in a topological vector space and let $F: X\times Y\to R$ be a function satisfying :

(1) For each $y\in Y$, $F(x,y)$ is upper semicontinuous and concave on X;

(2) for each $x\in X$, $F(x,y)$ is lower semicontinuous and convex on Y.

Then, we have

$$\sup_{x\in X}\inf_{y\in Y} F(x,y)=\inf_{y\in Y}\sup_{x\in X} F(x,y).$$

PROOF. Suppose that there exists a constant c such that

$$\sup_{y\in Y}\inf_{x\in X} F(x,y)<c<\inf_{y\in Y}\sup_{x\in X} F(x,y).$$

Then by Lemma 1, there exists a continuous mapping p of X in to Y such that $F(x,p(x))<c$ for all $x\in X$ and by Lemma 2, there exists a continuous mapping q of Y into X such that $F(q(y),y)>c$ for all $y\in Y$.

Let $Z=X\times Y$ and define $h: Z\times Z\to R$ by

$$\begin{aligned} &h((x,y)\ (q(y),\ q(x)))\\ &=(F(q(y),y)-c)\cap(c-F(x,p(x))) \end{aligned}$$

for all $(x,y)\in X\times Y$. Then we see that

$$\begin{aligned} h((x,y),\ (x,y))&=(F(x,y)-c)\cap(c-F(x,y))\\ &=\{0\}. \end{aligned}$$

Thus we have

$$F(x,y)=c$$

for all $(x, y) \in X \times Y$. This is a contradictabove. Consequently we have

$$\sup_{x \in X} \inf_{y \in Y} F(x, y) = \inf_{y \in Y} \sup_{x \in X} F(x, y).$$

LEMMA 3. Let X be a compact subset of a topological vector space E, Y be a convex subset of a topological vector space E and let F be a real valued function on $X \times Y$ satisfying:

(1) For each $y \in Y$, the function $F(x, y)$ of x is upper semicontinuous;

(2) for each $x \in X$, the function $F(x, y)$ of y is quasi-convex;

(3) for any constant c, $\sup_{x \in X} \inf_{y \in Y} F(x, y) < c$.

Then, there exists a continuous mapping p of X into Y such that $F(x, p(x)) < c$ for all $x \in X$.

PROOF. The proof is similar of Lemma 1. But the property of (2) we see that

$$\begin{aligned} F(x, p(x)) &= F\Big(x, \sum_{i=1}^{n} \beta_i(x) y_i\Big) \\ &\leqq \max\{F(x, y_i)\} < c \end{aligned}$$

for all $x \in X$. Thus, there exists a continuous mapping p of X into Y such that $F(x, p(x)) < c$ for all $x \in X$.

The proves of Lemma 4 and Theorem 2 are same method of Lemma 2 and Theorem 1 respectively.

LEMMA 4. Let Y be a compact subset of a topological vector space E, X be a convex subset of a topological vector space E and let F be a real valued function on $X \times Y$ satisfying:

(1) For each $y \in Y$, the function $F(x,y)$of x is quasi-concave;

(2) for each $x \in X$, the function $F(x,y)$ of y lower semicontinuous ;

(3) for any constant c, $\inf_{y \in Y} \sup_{x \in X} F(x,y) < c$.

Then, there exists a continuous mapping q of Y into X such that $F(q(y),y) > c$ for all $y \in Y$.

THEOREM 2. Let X and Y be compact convex subset each in a topological vector space and let $F: X \times Y \to R$ be a function satisfying :

(1) For each $y \in Y$, $F(x,y)$ is upper semicontinuous and quasiconcave on X;

(2) for each $x \in X$, $F(x,y)$ is lower semicontinuous and quasiconvex on Y.

Then, we have

$$\sup_{x \in X} \inf_{y \in Y} F(x,y) = \inf_{y \in Y} \sup_{x \in X} F(x,y).$$

DEFINITION 1. Let D denote an open bounded set of R^n, ∂D its boundary, f a mappin from $\bar{D}$ into R^n, and $a \in R^n - f(\partial D)$. If f is a $C'(D)$-mapping and $C^{\circ}(\bar{D})$-mapping,

$$\deg(f, D, a) = \sum_{x \in f^{-1}(a)} \operatorname{sign} J_f(x),$$

if

$$f^{-1}(a) \cap Z = \phi, \text{ with } Z = \{x \times J_f(x) = 0\},$$

Here, $J_f(x)$ is the Jacobian of f at the point x.

DEFINITION 2. Let $\{f_x\}$ denote a family of convex functions on R^n, depending on the parameter $x \in \overline{\Omega}$, where Ω is an open bounded subset of R^n. If there exists a continuous mapping

$$p: \partial\Omega \to R^n - \{0\}$$

and $\varepsilon \geqq 0$, such that

$$f_x(p(x)) < f_x(0) - \varepsilon, \quad \forall x \in \partial\Omega,$$

then we define

$$\deg(f_x) = \deg(p, \Omega, 0),$$

where p is a continuous extension of p to Ω.

THEOREM 3. Let $\{f_x\}$ be a family of closed convex functions, depending continuously on the parameter $x \in \overline{\Omega}$, where Ω is an open bounded subset of R^n. If $0 \notin \partial f_x(0)$ for all $x \in \partial\Omega$ and $\deg(f_x) \neq 0$, then there exists a $x_0 \in \Omega$, such that

$$f_{x_0}(0) = \inf_{y \in R^n} f_{x_0}(y).$$

PROOF. Assume that

$$\deg(f_x) \neq 0 \text{ and } f_x(0) > \inf_y f_x(y),$$

for all $x \in \Omega$. Since $0 \notin \partial f_x(0)$ for every $x \in \partial\Omega$, we have $f_x(0) > \inf_y f_x(y)$ for all $x \in \overline{\Omega}$. We denote the lower semicontinuous function $g(x) = f_x(0) - \inf_y f_x(y) > 0$.

Since

$\overline{\Omega}$ is compact, there exists a $x_0 \in \overline{\Omega}$ such that $g(x_0) =$

$\inf_{x \in \bar{\Omega}} g(x) > 0$. We take $\varepsilon (0 < \varepsilon < g(x_0))$. Then we have

$$f_x(0) > \inf_y f_x(y) + \varepsilon$$

for all $x \in \bar{\Omega}$, which implies that

$$\sup_{x \in \bar{\Omega}} \inf_y (f_x(y) - f_x(0)) < -\varepsilon.$$

By Lemma 1, there exists a continuous mapping $p : \bar{\Omega} \to R^n - \{0\}$ such that

$$f_x(p(x)) < f_x(0) - \varepsilon.$$

By [2]

$$\deg(f_x) = \deg(p, \Omega, 0) = 0$$

becaused $0 \notin p(\bar{\Omega})$. This is a contraction Thus theorem is complete.

DEFINITION 3. Let (E, F) be a dual system. and $f: E \to \bar{R}$ is a quasiconvex mapping. For any $x_0 \in E$, the quasisubgradient of f at x_0 is the set $\partial^* f(x_0) \subset F$, defined by

$$x^* \in \partial^* f(x_0) \text{ if } (x^*, x - x_0) \geqq 0 \text{ then } f(x) \geqq f(x_0).$$

THEOREM 4. Let $\{f_x\}$ be a family of closed quasiconvex functions, depending continuously on the parameter $x \in \bar{\Omega}$, where Ω is an open bounded subset of R^n. If $0 \notin \partial^* f_x(0)$ for all $x \in \partial\Omega$ and $\deg(f_x) \neq 0$, then there exists a $x_0 \in \Omega$, such that

$$f_{x_0}(0) = \inf_{y \in R^n} f_{x_0}(y).$$

PROOF. By using of Lemma 3, [the method is same as Theorem 3.

Reference

1. F. E. Browder, On a new generalization of the Schauder fixed point theorem, Math. Ann., 174 (1967), 285-290.
2. J. T. Schwartz, Nonlinear Functional Analysis, Gordon and Breach, New York, New York, 1969.
3. J. Cronin, Fixed points and topological degree in nonlinear analysis, A. M. S., Providence, Rhode Island, 1964.
4. W. Takahashi, Nonlinear variational inequalities and fixed point theorems, J. Math. Soc. Japan, 28 (1976), 168-181.

Pusan National University
Pusan 609-735
Korea

Received May 6, 1988