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A STUDY ON SUBMANIFOLDS OF CODIMENSION 2 IN A SPHERE

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1. Introduction

Let M be an n-dimensional compact connected and oriented Riemannian manifold isometrically immersed in an $(n+2)$-dimensional Euclidean space R^{n+2}. Moore [5] proved that if M is of positive curvature, then M is a homotopy sphere. This result is generalized by Baldin and Mercuri [2], Baik and Shin [1] to the case of non-negative curvature, which is stated as follows: If M is of non-negative curvature, then M is either a homotopy sphere or diffeomorphic to a product of two spheres. In particular, if there is a point at which the curvature operator is positive, then M is homeomorphic to a sphere.

The purpose of this paper is to verify the following:

THEOREM A. *Let M be an $n(\geq 3)$-dimensional compact connected and oriented Riemannian manifold isometrically immersed in an $(n+2)$-dimensional sphere $S^{n+2}(c)$. If all sectional curvature of M are not less than the positive constant c, then M is a real homology sphere.*

2. Lemmas

Let V and W be real vector spaces of finite dimensions n and p respectively, and $B : V\times V\to W$ a symmetric bilinear form on V with values in W. Suppose $n\geq 2$ and W has an inner product $\langle\, ,\, \rangle$. Define the associated curvature form $R_B : \Lambda^2 V\times\Lambda^2 V\to R$ by

(2.1) $R_B(x\wedge y, z\wedge w)=\langle B(x, z), B(y, w)\rangle-\langle B(x, w), B(y, z)\rangle$.

Then R_B is again symmetric, and positive semi-definite iff $R_B(w, w)\geq 0$ whenever $w\neq 0$. Let K_B be the associated sectional curvature form

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defined by

(2.2) $$K_B(x, y)=R_B(x\wedge y, x\wedge y).$$

We say that K_B has positive semi-definite iff $K_B(x, y)\geq 0$ whenever $x\wedge y\neq 0$.

Consider the following conditions on B:

(a) There exists an orthonormal basis $\{e_1, \cdots, e_p\}$ for W such that real valued forms on V defined by $(x, y)\rightarrow\langle B(x, y), e_i\rangle$ are all non-negative for the indices $i=1, \ldots, p$.

(b) R_B is positive semi-definite.

(c) K_B is positive semi-definite.

The following Lemma was proved by Baik and Shin [1].

LEMMA 1. *(a)→(b)→(c). In particular, if $p=2$, then the conditions are all equivalent.*

Next, we define the curvature operator in a Riemannian manifold M. For a point x of M let R_x be an associated curvature form. A linear map ρ_x^* of Λ^2M_x into $\Lambda^2M_x^*$ is defined by $u\wedge v\rightarrow R_x(\cdot, \cdot, u, v)$, where M_x^* denotes the dual space of the tangent space M_x at x. By this dual endomorphism, ρ_x of $\Lambda^2M_x^*$ into itself is constructed. It turns out that ρ_x satisfies

(2.3) $$\rho_x(u^*\wedge v^*)(w^*\wedge z^*)=\rho_x^*(u\wedge v)(w^*\wedge z^*)=R_x(u, v, w, z),$$

for any vectors u, v, w and z in M_x, where u^* denotes the dual form in M_x^* associated with the vector u. The operator ρ_x is called a curvature operator at x of M. Since ρ_x is a symmetric operator, each eigenvalue of it is real. If all eigenvalues of ρ_x are contained in the closed interval $[\lambda, \Lambda]$, then one says $\lambda\leq\rho_x\leq\Lambda$, and if for any point x of M this property is satisfied, then $\rho(M)$ is said to satisfy the condition $\lambda\leq\rho(M)\leq\Lambda$, where $\rho(M)$ is the set consisting of all curvature operators at all points of M.

The following Lemma is due to Meyer [4].

LEMMA 2. *Let M be an n-dimensional compact and oriented Riemannian manifold. If all curvature operators satisfy the condition $\rho(M)>0$, then M is a homology sphere.*

3. Proof of Theorem A

Let i be an isometric immersion of M into the sphere $\overline{M}=S^{n+2}(c)$. For any point x of M we shall denote $i(x)$ on $\overline{M}$ by the same symbol x, since there is no danger of confusion and, moreover, the computation is local. Furthermore, a tangent vector u at x is identified with $di_x(u)$. Then the tangent space M_x at x is a subspace of the tangent space $\overline{M}_x$ of ambient space $\overline{M}$ at x. Let N_x be the orthogonal complement of M_x in $\overline{M}_x$, called a normal space to M at x, and let h be the second fundamental form of the immersion i.

Let R_B be the associated curvature form on M_x which is defined by (2. 1) and K_B be the real valued map on $M_x \times M_x$ defined by (2. 2). From (2. 1) we get

(3. 1) $R_B(u\wedge v, w\wedge z)=R(u, v, w, z)-c(\langle u, w\rangle\langle v, z\rangle-\langle u, z\rangle\langle v, w\rangle)$,

where R denotes the Riemannian curvature tensor on M. Then we have

(3. 2) $K_B(u, v)=(K(u, v)-c)(\|u\|^2\|v\|^2-\langle u, v\rangle^2)$,

where $K(u, v)$ is the sectional curvature of plane spaned by linearly independent vectors u and v on M_x. By the assumption of the Theorem it follows that $K_B \geq 0$. Thus, by the Lemma 1, the associated curvature form R_B satisfies $R_B \geq 0$. Hence the curvature operator ρ_x at x of M satisfies $\rho_x \geq c$ because of (2. 3). Then we have $\rho(M) \geq c > 0$. By the Lemma 2, M is a real homology sphere. This concludes the proof.

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