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ISOTROPY REPRESENTATIONS OF CYCLIC GROUP ACTIONS ON HOMOTOPY SPHERES

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1. Introduction

Let Σ be a smooth compact manifold without boundary having the same homotopy type as a sphere, which is called a homotopy sphere. Suppose a group G acts smoothly on Σ with the fixed point set Σ^G consists of two isolated fixed points p and q. In this case, tangent spaces $T_p\Sigma$ and $T_q\Sigma$ at isolated fixed points, as isotropy representations of G are called Smith equivalent. Moreover Σ is called a supporting homotopy sphere of Smith equivalent representations $T_p\Sigma$ and $T_q\Sigma$.

The study on Smith equivalence has rich history, and for this we refer the reader to [P] or [Su]. The following question of P. A. Smith [S] motivates the study on Smith equivalence.

QUESTION. Does Smith equivalence imply equivalence of representations?

In this paper we consider the question for cyclic group of order $2d$ where d is odd. Let $G=Z_{2d}$ be the cyclic group of order $2d$ where d is odd. Let H be the index 2 subgroup. Thus H is the cyclic group of odd order. The main results are as follows:

THEOREM A. *Suppose V and W are Smith equivalent representations of G. If H is an isotropy subgroup of either V or W, then V and W are equivalent.*

An easy corollary is as follows:

COROLLARY B. *Any odd dimensional homotopy sphere can not support non equivalent Smith equivalent representations.*

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Note that if $G=Z_{4n}$ with $n\geq 2$, Cappell-Shaneson [CS], Petrie [P], and Dovermann [D] show that there exist non-equivalent Smith equivalent representations. There the index 2 subgroup H appears as an isotropy subgroup and such fact is used in essential ways.

2. Proof of Main Results

Suppose V and W are Smith equivalent representations of G. Let the index 2 subgroup H be an isotropy subgroup of one representation, say V. Let Σ be a supporting homotopy sphere of Smith equivalent representations V and W. Consider the fixed point set $\Sigma^H=\{x\in\Sigma \mid hx=x \text{ for all } h\in H\}$. Then $G/H\cong Z_2$ act smoothly on Σ^H with the fixed point set $(\Sigma^H)^{G/H}=\Sigma^G=\{p, q\}$.

We claim that p and q are in a same connected component of Σ^H. Suppose not. Let the connected component of Σ^H containing p be $X(p)$. Then $X(p)$ is a Z_2 smooth manifold with the fixed point set $X(p)^{Z_2}=\{p\}$. Let

$$f: X(p) \longrightarrow S(T_pX(p)\oplus R)=Y$$

be the map which collapses outside of small neighborhood of p in $X(p)$ to a point.

Then f is a Z_2 map of degree 1. At this point we need the following result of Bredon [Theorem 5.1, B]

LEMMA 2.1. *Let X and Y be smooth orientable manifolds with Z_p action for prime p. Let $f: X\rightarrow Y$ be an Z_p equivariant map of non-zero degree (mod p). Then the induced homomorphism*

$$H^*(Y^G, Z_p) \longrightarrow H^*(X^G, Z_p)$$

is a monomorphism.

From lemma 2.1

$$f^*: H^0(Y^{Z_2}, Z_2) \longrightarrow H^0(X(p)^{Z_2}, Z_2)$$

is a monomorhpism. This is impossible because Y^{Z_2} consists of two points while $X(p)^{Z_2}$ consists of a single point. We thus have proved that p and q are in a same connected component of Σ^H.

For any representation V of G and any subgroup K of G, $res_K V$ is the representation V restricted to K. Since p and q are in the same connected component of $\sum^H$ representation $res_H V$ and $res_H W$ are equivalent. For the next step we need the following classical result of Smith.

LEMMA 2. 2. *If G is a p-group and G acts on a homology sphere X, then $\sum^G$ is a Z_p homology sphere.*

From lemma 2. 2 $\sum^{Z_2}$ is a Z_2 homology sphere. In particular, $\sum^{Z_2}$ is connected. Thus

$$V^{Z_2}=(T_p\sum)^{Z_2}=T_p(\sum^{Z_2})=T_q(\sum^{Z_2})=(T_q\sum)^{Z_2}=W^{Z_2}$$

as a representation of $H\cong G/Z_2$. The following lemma proves theorem A.

LEMMA 2. 3. *Let V and W be two real representations of $G=Z_{2d}$, d is odd, such that $V^G=W^G=0$. If $res_H V=res_H W$ and $V^{Z_2}=W^{Z_2}$, then $V=W$.*

Proof. Let t^i be a complex 1-dimensional repesentation of $G=Z_{2d}=\{g^k\,|\,k=0, 1, \cdots, 2d-1,\ g=\exp 2\pi/2d\}$ such that $g^k(z)=g^{ki}\cdot z$ for $z\in t^i=C$. Let $\tilde{t}^i$ be the realification of t^i. Then $\tilde{t}^i$ is an irreducible real representation of G if $i\neq 0$, d. It is easy to see that $\tilde{t}^i=\tilde{t}^{2d-i}$ for all i. Let R_- be the irreducible real 1-dimensional representation of G such that $g^k\cdot x=(-1)^k x$ for $g^k\in G, t\in R_-$. Then $\{\tilde{t}^1, \cdots, \tilde{t}^{d-1}, R_-\}$ is the set of all irreducible real representations of G and any real representation V with $V^G=0$ can be written $V=\sum_{i=1}^{d-1}a_i\tilde{t}^i+a_dR_-$.

Let

$$V=\sum_{i=1}^{d-1}a_i\tilde{t}^i+a_dR_-$$

$$W=\sum_{i=1}^{d-1}b_i\tilde{t}^i+b_dR_-$$

Since $res_H V=res_H W$ $a_i+a_{d-i}=b_i+b_{d-i}$ for all $i=1, \cdots, d-1$ and $a_d=b_d$. On the other hand

$$V^{Z_2}=a_2\tilde{t}^2+a_4\tilde{t}^4+\cdots+a_{d-1}\tilde{t}^{d-1}$$

$$W^{Z_2}=b_2\tilde{t}^2+b_4\tilde{t}^4+\cdots+b_{d-1}\tilde{t}^{d-1}$$

Since $V^{Z_2}=W^{Z_2}$ as a G/Z_2 representation $a_{2k}=b_{2k}$ for all $k=1, \cdots, (d-1)/2$ $a_i=b_i$ for all $i=1, \cdots, d$, which proves the lemma.

Proof of Corollary B. If $\sum$ is an odd dimensional homotopy sphere supporting Smith equivalent representations V and W, $\dim V=\dim W$

is odd. Since V and W can not contain the trivial representation as a subrepresentation V and W must contain R_- as an irreducible subrepresentaion. But if this happens, clearly H is an isotropy subgroup of V and W. Thus the corollary follows from theorem A.

References

B. G. Bredon, *Fixed point set of actions on Poincaré duality spaces*, Topology **12** (1973), 159-175.

CS. S. Cappell and J. Shaneson, *Fixed points of periodic differentiable maps*, Invent. Math. **68** (1982).

D. K.H. Dovermann, *Even dimensional Smith equivalent representations*, Algebraic Topology Conf. Aarhus, Springer-Verlag LNM **1051** (1982).

P. T. Petrie, *Smith equivalence of representations*, Proc. Cambridge Philo. Soc. **94** (1983).

S. P.A. Smith, *New results and old problems in finite transformation groups*, Bull. AMS **66** (1960), 401-415.

Su. D.Y. Suh, *Smith equivalence of representations, Thesis*, Rutgers Universiyty (1984).

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