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FIXED POINT ALGEBRAS OF UHF-ALGEBRAS*

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1. Introduction

In this paper we study a C^*-dynamical system (A, G, α) where A is a UHF-algebra, G is a finite abelian group and α is a $*$-automorphic action of product type of G on A. In [2], A. Kishimoto considered the case $G=Z_n$, the cyclic group of order n and investigated a condition in order that the fixed point algebra A^α of A under the action α is UHF. In later N. J. Munch studied extremal tracial states on A^α by employing the method of A. Kishimoto [3], where G is a finite abelian group. Generally speaking, when G is compact (not necessarily discrete and abelian), A^α is an AF-algebra and its ideal structure was well analysed by N. Riedel [4].

Here we obtain some conditions for A^α to be UHF, where G is a finite abelian group, which is an extension of the result of A. Kishimoto.

2. Notations and preliminaries

Let G be a finite abelian group and K_n, $n\in \boldsymbol{N}$ be matrix factors of rank $|K_n|$. Consider unitary representations $\pi_n : G\to K_n$ and define the homomorphism α of G into the group of all $*$-automorphisms of $A=\bigotimes_{n=1}^{\infty} K_n$ by $\alpha_g=\bigotimes_{n=1}^{\infty} Ad\pi_n(g)$.

Put $W_g^{n,m}=\bigotimes_{i=n}^{m}\pi_i(g)$, $n\leqq m$. Since $W^{n,m}$ is a unitary representation of G into $\bigotimes_{i=n}^{m} K_i$, we obtain a spectral decomposition $W_g^{n,m}=\sum_{\mu\in\hat{G}}\mu(g)E_\mu^{n,m}$, where $E_\mu^{n,m}$ are projections in $\bigotimes_{i=n}^{m} K_i$ and $\hat{G}$ is the character group

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of G, considering the irreducible decomposition of $W^{n,m}$. Note that $E_\mu^{n,m}=\sum_{\nu\in\hat{G}} E_{\mu-\nu}^{n,s}\otimes E_\nu^{s,m}$ for $n\leqq s\leqq m$ by uniqueness of irreducible decomposition. Also note that $E_\mu^{n,m}=|G|^{-1}\sum_{g\in G}\mu(g)W_g^{n,m}$, where $|G|$ denotes the order of G.

We assume that α_g is outer whenever $g\neq e$, the identity of G. Also we may assume all $E_\mu^{m,n}$ to be non zero as same as in [3]. By [1, 3, 4], $A^\alpha=\{x\in A\ ;\ \alpha_g(x)=x$ for all $g\in G\}$ is equal to $(\bigcup_{n=1}^{\infty} A_n^\alpha)^-$ where $A_n=\bigotimes_{i=1}^{n} K_i$ and $^-$ denotes the norm closure. Put $A_\mu^n=E_\mu^{1,n}\ A_nE_\mu^{1,n}$ for $\mu\in\hat{G}$. Then A_μ^n is a matrix factor and $A_n^\alpha=\sum_{\mu\in\hat{G}}\oplus A_\mu^n$. Therefore A^α is an AF-algebra (See [1]).

Let τ be a canonical trace on the UHF-algebra A, that is, $\tau=\bigotimes_{i=1}^{\infty}|K_i|^{-1}Tr$, where Tr is the usual trace on the matrix algebra K_i. Describing the structure of the AF-algebra A^α, we must know how A_ρ^n is partially embedded into A_μ^{n+1} for $\rho,\mu\in\hat{G}$. By [3, Lemma 2.1] its multiplicity is

$$|K_{n+1}|\tau(E_{\mu-\rho}^{n+1,n+1}). \tag{2.1}$$

Let $B(l^2(G))$ be the algebra of all operators on $l^2(G)$. We define a regular representation λ of G on $l^2(G)$ by

$$(\lambda(g)\xi)(h)=\xi(h-g) \text{ for } g,h\in G,\ \xi\in l^2(G).$$

B denotes the UHF-algebra $\bigotimes_{n=1}^{\infty}B(l^2(G))$, i.e., the infinite tensor product of copies of $B(l^2(G))$ with type $|G|^\infty$. Also we define the action β of G on B such that $\beta_g=\bigotimes_{n=1}^{\infty}Ad\lambda(g)$ for all $g\in G$.

3. Main result

LEMMA. *The fixed point algebra B^β is $*$-isomorphic to B.*

Proof. By (2.1), we know the multiplicity of partial embedding of B_ρ^n into B_μ^{n+1} for $\rho,\mu\in\hat{G}$ as follows,

$$\begin{aligned}|B(l^2(G))|\tau(E_{\mu-\rho}^{n+1,n+1})&=Tr(|G|^{-1}\sum_{g\in G}(\mu-\rho)(g)\lambda(g))\\&=|G|^{-1}\sum_{g\in G}\mu(g)\overline{\rho(g)}Tr(\lambda(g))=1\end{aligned}$$

since $Tr(\lambda(g))=|G|\delta_{g,e}$, where δ is the Kronecker's delta. Hence the Bratteli diagram for B^{β} is isomorphic to that of B [1].

We give a condition for the fixed point algebra A^{α} of a UHF-algebra A by the product type action α of a finite abelian group G to be UHF, which is a generalization of the result of A. Kishimoto [2].

THEOREM. *Let (A, G, α) be a C^*-dynamical system as in section 2. Then the followings are equivalent:*

(i) *A^{α} is a UHF-algebra.*

(ii) *A^{α} is $*$-isomorphic to A.*

(iii) *A is $*$-isomorphic to $C\otimes B$ for some UHF-algebra C and α is conjugate to $id\otimes\beta$, where id is the identity automorphism of C.*

(iv) *There exists an increasing sequence $\{n_k : k=1, 2, \dots\}$ of non negative integers such that $n_1=0$ and*

$$\tau(E_{\mu-\rho}^{n_k+1, n_{k+1}})=|G|^{-1} \text{ for all } \mu, \rho\in\hat{G}, \text{ all } k.$$

Proof. By the above lemma, implications (iii)→(ii)→(i) are obvious.

We will prove that (i) implies (iv). Since $A^{\alpha}=(\bigcup_{n=1}^{\infty} A_n^{\alpha})^-$ is a UHF-algebra, there exist an increasing sequence $\{B(k) : k=1, 2, \dots\}$ of matrix factors and $\{n_k : k=1, 2, \dots\}$ of non negative integers such that

$$A_{n_k}^{\alpha}\subset B(k)\subset A_{n_{k+1}}^{\alpha}$$

(see [1] 2.5). Let a_{ρ}^k (resp. b_{ρ}^k) be the multiplicity of partial embedding of $A_{\rho}^{n_k}$ into $B(k)$ (resp. $B(k)$ into $A_{\rho}^{n_{k+1}}$). Then the multiplicicity of partial embedding of $A_{\rho}^{n_k}$ into $A_{\mu}^{n_{k+1}}$ is $a_{\rho}^k b_{\mu}^k$ and

$$a_{\rho}^k b_{\mu}^k=\prod_{i=n_k+1}^{n_{k+1}}|K_i|\tau(E_{\mu-\rho}^{n_k+1, n_{k+1}}). \tag{3.1}$$

Now we have

$$\begin{aligned}\sum_{\rho\in\hat{G}} a_{\rho}^k b_{\mu}^k &=\prod_{i=n_k+1}^{n_{k+1}}|K_i|\sum_{\rho\in\hat{G}}\tau\Big(\sum_{g\in G}|G|^{-1}(\mu-\rho)(g)\,W_g^{n_k+1, n_{k+1}}\Big)\\ &=\prod_{i=n_k+1}^{n_{k+1}}|K_i|\,|G|^{-1}\sum_{g\in G}\tau(W_g^{n_k+1, n_{k+1}})\mu(g)\sum_{\rho\in\hat{G}}\rho(g)\\ &=\prod_{i=n_k+1}^{n_{k+1}}|K_i|\,|G|^{-1}\sum_{g\in G}\tau(W_g^{n_k+1, n_{k+1}})\mu(g)\,|G|\delta_{g,e}\\ &=\prod_{i=n_k+1}^{n_{k-1}}|K_i|.\end{aligned}$$

Similarly $\sum_{\mu\in\hat{G}} a_\rho^k b_\mu^k = \prod_{i=n_k+1}^{n_{k+1}} |K_i|$. Hence a_ρ^k (resp. b_μ^k) is independent for $\rho\in\hat{G}$ (resp. $\mu\in\hat{G}$). We put $a_\rho^k = a_k$ ($b_\mu^k = b_k$) for all $\rho\in\hat{G}$ ($\mu\in\hat{G}$). Therefore we have $a_k b_k = \prod_{i=n_k+1}^{n_{k+1}} |K_i|\,|G|^{-1}$ i. e., $\tau(E_{\mu-\rho}^{n_k+1,n_{k+1}}) = |G|^{-1}$ for all $\mu, \rho\in\hat{G}$ since of (3. 1), proving (iv).

Next we suppose (iv). Then we have for all $\mu\in\hat{G}$, $\tau(E_\mu^{n_k+1,n_{k+1}}) = |G|^{-1}$, which implies for all $g\in G$

$$\tau(W_g^{n_k+1,n_{k+1}}) = \tau(\sum_{\mu\in\hat{G}}\mu(g)E_\mu^{n_k+1,n_{k+1}}) = |G|^{-1}\sum_{\mu\in\hat{G}}\mu(g) = \delta_{g,e}.$$

Hence we know the character's identity

$$Tr(W_g^{n_k+1,n_{k+1}}) = \prod_{i=n_k+1}^{n_{k+1}} |K_i|\,|G|^{-1} Tr(\lambda_g), \tag{3.2}$$

where Tr denotes the usual trace of $\bigotimes_{i=n_k+1}^{n_{k+1}} K_i$ or $B(l^2(G))$. C_k denotes the matrix factor of dimension $\prod_{i=n_k+1}^{n_{k+1}} |K_i|\,|G|^{-1}$. By (3. 2) there exists a $*$-isomorphism ϕ_k of $\bigotimes_{i=n_k+1}^{n_{k+1}} K_i$ onto $C_k\otimes B(l^2(G))$ and $Ad\,W_g^{n_k+1,n_{k+1}} = \phi_k^{-1}(id\otimes Ad\lambda_g)\phi_k$ for all $g\in G$ where id is the identity automorphism of C_k. Put $C = \bigotimes_{k=1}^{\infty} C_k$, a UHF-algebra and $\phi = \bigotimes_{k=1}^{\infty}\phi_k$, a $*$-isomorphism of A onto $C\otimes B$ (identifying $C\otimes B$ and $\bigotimes_{k=1}^{\infty}(C_k\otimes B(l^2(G)))$). Then we have $\alpha_g = \phi^{-1}(id\otimes\beta_g)\phi$ for all $g\in G$, which proves (iii).

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