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PROPERTIES OF CAUSALLY CONTINUOUS SPACE-TIME

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1. Introduction

In general relativity, analyzing causality is central to the study of black holes, to cosmology, and to each of the major recent mathematical theorems. By causality we refer to the general question of which points in a space-time can be joined by causal curves; relativistically which events can influence (be influenced by) a given event. Various causality conditions have been developed for space-times of the problems associated with examples of causality violations (2, 4).

Causally continuous space-times were defined by Hawking and Sachs (5). Budic and Sachs (3) established causal completion. A metrizable topology on the causal completion of a causally continuous space-time was studied by Beem (1). Recently the region of space-time where causal continuity is violated was studied by Ishikawa (6) and Vyas and Akolia (8).

In this paper we show characterization for reflectingness in terms of continuity of set valued functions. We investigate some properties of the region related to a causally continuous space-time where distinguishingness is violated, and characterize the chronology condition in terms of distinguishing-violated region.

2. Preliminaries

Let M be a space-time. This means that M is a connected C^∞ Hausdorff manifold of dimension ≥ 2 which has a countable basis, a Lorentzian metric g of signature $(-, +, \ldots, +)$, and a time-orientation. If there is a future-directed timelike curve from p to q, we write $p \ll q$. The chronological future $I^+(p)$ and past $I^-(p)$ are defined by

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$$I^+(p)=\{q\mid p\ll q\} \text{ and } I^-(p)=\{q\mid q\ll p\}.$$

For all p in M, $I^+(p)$ and $I^+(p)$ are open (4). The chronological common future $\uparrow S$ of an open set S of M is defined by

$$\uparrow S=I^+\{x\in M\mid s\ll x \text{ for all } s\in S\}.$$

Dually the chronological common past $\downarrow S$ is defined.

A space-time (M, g) is chronological if it contains no closed timelike curves. M is future reflecting if $I^-(x)\subseteq I^-(y)$ implies $I^+(y)\subseteq I^+(x)$, and it is past reflecting if $I^+(x)\subseteq I^+(y)$ implies $I^-(y)\subseteq I^-(x)$. M is reflecting if it is both past and future reflecting. A space-time (M, g) is future (past) distinguishing if $I^+(x)=I^+(y)$ $(I^-(x)=I^-(y))$ implies $x=y$. M is distinguishing if it is future and past distinguishing. We say that M is strongly causal if for every p in M and any neighborhood U_p of p, there is a neighborhood V_p of p contained in U_p such that no nonspacelike curve which leaves V_p ever return to it. M is said to be causally continuous if it is distinguishing and reflecting. All of the other terminologies will be referred to (2, 4).

3. Main Results

Let F be a function which assigns to each z in M an open set $F(z)$ in M. F is called inner continuous if for any z and any compact set $K\subseteq F(z)$, there is a neighborhood U of z such that $K\subseteq F(x)$ for all x in U. Also F is outer continuous if for any z and any compact set $K\subseteq M-cl\,F(z)$, there exists a neighborhood U of z such that $K\subseteq M-cl\,F(x)$ for all x in U, where cl denotes closure operator.

The following Lemmas were shown in (5);

LEMMA 1. *I^+ and I^- are always inner continuous.*

LEMMA 2. *The following conditions are equivalent.*

(a) *M is future (respectively, past) reflecting.*

(b) *If x is an element of $cl\,I^-(y)$, then y belongs to $cl\,I^+(x)$ (respectively, if x is an element of $cl\,I^+(y)$, then y belongs to $cl(I^-(x))$.*

(c) *For all z in M, $\uparrow I^-(z)=I^+(z)$ (respectively $\downarrow I^+(z)=I^-(z)$).*

The set valued functions $\uparrow I^-$ and $\downarrow I^+$ characterize reflectingness as follows;

LEMMA 3. *A space-time* (M, g) *is future (past) reflecting if and only if* $\uparrow I^-$ $(\downarrow I^+)$ *is inner continuous.*

Proof. Suppose that (M, g) is future reflecting. Then $\uparrow I^- = I^+$ on M by Lemma 2 and $\uparrow I^-$ is inner continuous by Lemma 1.
Conversely, suppose that $\uparrow I^-$ is inner continuous and z in M. If y is an element of $I^-(z)$, there is a neighborhood U of z such that y belongs to $\uparrow I^-(x)$ for all x in U. Taking x in U with $z \ll x$, we have $z \ll y$. Hence y belongs to $I^+(z)$.
Thus $\uparrow I^-(z) \subseteq I^+(z)$.
On the other hand $\uparrow I^-(z) \supseteq I^+(z)$. Therefore we have $\uparrow I^-(z) = I^+(z)$ for all z in M. By Lemma 2, (M, g) is future reflecting.
Thus by using the above Lemma 3 and Theorem 2.1 in (5), we have the following theorem;

THEOREM 4. *The following conditions are equivalent;*
(a) M *is reflecting.*
(b) I^+ *and* I^- *are outer continuous.*
(c) $\uparrow I^-$ *and* $\downarrow I^+$ *are inner continuous.*

From these facts, we know that causally continuous space-times are those distinguishing space-times for which the chronological future of a point and its chronological commom past, and the chronological past of a point and its chronological common future vary continuously with the point.

We now study some properties of the region of a space-time M where distinguishingness fails. We first define

$$\tilde{D}^+(x) = \{y \in M \mid I^+(x) = I^+(y)\},$$
$$\tilde{D}^-(x) = \{y \in M \mid I^-(x) = I^-(y)\},$$
$$D^+(x) = \tilde{D}^+(x) - \{x\} \text{ and } D^-(x) = \tilde{D}^-(x) - \{x\}.$$

The sets D^+ and D^- are the sets of points where future and past distinguishing conditions are violated respectively. And we denote D is the union of D^+ and D^-.

THEOREM 5. (a) *y belongs to $\tilde{D}^+(x)$ if and only if $cl\, I^+(x) = cl\, I^+(y)$.*
(b) *$\tilde{D}^+(x)$ for x in D^+ forms a partition of D^+.*

Proof. (a) y belongs to $cl\, I^+(x)$ if and only if
$I^+(y) \subseteq I^+(x)$ (see (3)).
(b) Define $x \sim y$ in D^+ if and only if y belongs to $\tilde{D}^+(x)$. Then the relation $\sim$ is an equivalence one on D^+ with $\tilde{D}^+(x)$ equivalence class.

THEOREM 6. *If (M, g) is a reflecting space-time, then*
(a) *$\tilde{D}^+(x) = \tilde{D}^-(x) = cl\, I^+(x) \cap cl\, I^-(x)$ for all x in M.*
(b) *$D^+ = D^- = D$, and D is a closed set.*

Proof. (a) Since (M, g) is reflecting, z belongs to $\tilde{D}^+(x)$ if and only if $I^+(x) = I^+(z)$ if and only if $I^-(x) = I^-(z)$ if and only if z belongs to $\tilde{D}^-(x)$. Thus $\tilde{D}^+(x) = \tilde{D}^-(x)$.
On the other hand, z belongs to $cl\, I^+(x) \cap cl\, I^-(x)$ if and only if $I^+(z) \subseteq I^+(x)$ and $I^+(z) \subseteq I^-(x)$ if and only if $I^-(z) \supseteq I^-(x)$ and $I^-(z) \subseteq I^-(x)$ if and only if $I^-(z) = I^-(x)$ if and only if z belongs to $D^-(x)$. Thus $cl\, I^+(x) \cap cl\, I^-(x) = \tilde{D}^-(x)$.
(b) By (a), $D^+ = D^- = D$.
The set of points of M at which strong causality is violated is a closed subset of M. In reflecting space-times, distinguishingness and strong causality are equivalent. Thus D is a closed set.

A space-time (M, g) is said D-symmetric if violation of past and future distinguishingness occures simultaneously. In other words, $\tilde{D}^+(x) = \tilde{D}^-(x)$ for all x in M. A space-time (M, g) is locally reflecring if every point in M has a future and past reflecting neighborhood; that is, for any x in M, there exists a neighborhood U_x of x, such that if y belongs to U_x and z belongs to U_x, then

$$I^+(y) \subset I^+(z) \text{ if and only if } I^-(z) \subset I^-(y).$$

Every reflecting space-time is evidently D-symmetric and locally reflecting, but its converse does not hold in general.
For an example, see the following figure 1:

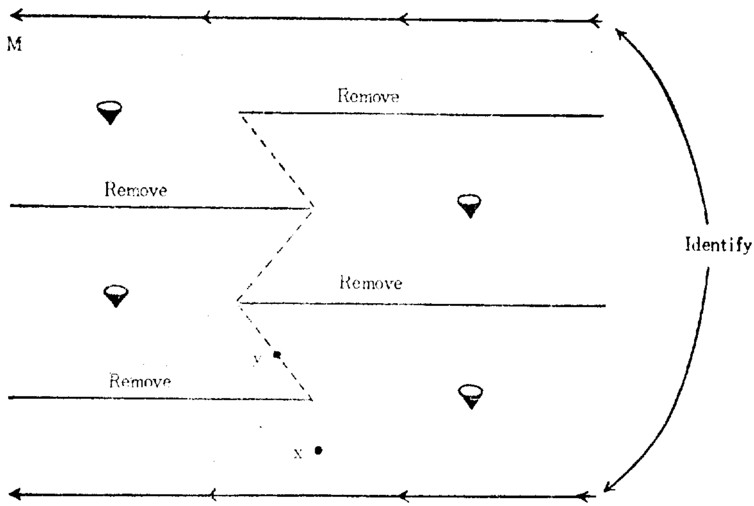

Figure 1. The space-time M is D-symmetric and locally reflecting, But M is not reflecting; $I^+(y) \subseteq I^+(x)$, but $I^-(x) \cap I^-(y) = \phi$.

THEOREM 7. *Let (M, g) be a D-symmetric and locally reflecting space-time. Then for all x in M, $\tilde{D}^+(x)$ is closed set.*

Proof. Suppose that z does not belong to $D^+(x)$.

Case 1. $I^+(z) - I^-(x) \neq \phi$: Let p belong to $I^+(z) - I^-(x)$. Then $I^-(p)$ contains z and $I^-(p) \cap \tilde{D}^+(x) = \phi$.

Case 2. $I^-(z) - I^-(x) \neq \phi$: Let p belong to $I^-(z) - I^-(x)$. Then z belongs to $I^+(p)$. If some q belongs to $I^+(p) \cap \tilde{D}^+(x)$, then $I^+(q) = I^+(x)$ and p belongs to $I^-(q)$. Since M is D-symmetric, $I^-(q) = I^-(x)$ and hence p belongs to $I^-(x)$. It contradicts. Therefore $I^+(p) \cap \tilde{D}^+(x) = \phi$.

Case 3. $I^+(z) \subsetneq I^+(x)$ and $I^-(z) \subsetneq I^-(x)$: Taking a reflecting neighborhood U of z, we have $U \cap \tilde{D}^+(x) = \phi$.
It follows that there is a neighborhood of z that is entirely contained in the complement of $\tilde{D}^+(x)$.

This proposition is not true in general. For example, see the following figure 2:

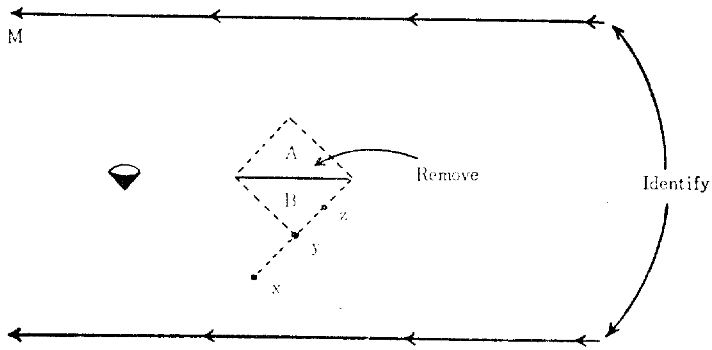

Figure 2. For x in M, $\tilde{D}^{\pm}(x)$ are not closed subsets of M; $\tilde{D}^{+}(x)=M-(A\cup cl\, B)$, $\tilde{D}^{-}(x)=M-(B\cup cl\, A)$. Note that the space-time M is not D-symmetric; z belongs to $\tilde{D}^{-}(x)$, but z does not belong to $\tilde{D}^{+}(x)$. Also, it is not locally reflecting; y has no reflecting neighborhood.

Now we prove that the chronology condition on a space-time can be characterized in terms of the distinguishing-violated region.

THEOREM 8. *The following conditions are equivalent.*
(a) (M, g) *is chronological.*
(b) *Interior of* $\tilde{D}^{+}(x)$ *is empty for all* x *in* M.
(c) *Interior of* $\tilde{D}^{-}(x)$ *is empty for all* x *in* M.

Proof. (a) $\Longleftrightarrow$ (b). Suppose that the condition (a) holds. If interior of $\tilde{D}^{+}(x)$ is not empty for some x in M, then interior of $\tilde{D}^{+}(x)$ is a neighborhood of some z. Thus $\tilde{D}^{+}(x)\cap I^{+}(z)\neq\phi$ and $I^{+}(z)=I^{+}(x)$. Hence there is a point p in M such that p belongs to $\tilde{D}^{+}(x)\cap I^{+}(x)$. Then p belongs to $I^{+}(p)$. It contradicts to (a).

Suppose that (b) holds. If M is not chronological, then there is p in M such that p belongs to $I^{+}(p)$. Since I^{+} is inner continuous by Lemma 1, there is a neighborhood U of p such that p belongs to $I^{+}(x)$ for all x in U. Since I^{-} is also inner continuous, there is a neighborhood V of p such that p belongs to $I^{-}(y)$ for all y in V. Thus $U\cap V$ is a neighborhood of p and p belongs to $I^{+}(z)\cap I^{-}(z)$ for all z in $U\cap V$. Therefore $I^{+}(p)=I^{+}(z)$ for all z in $U\cap V$. Hence $U\cap V\subseteq\tilde{D}^{+}(p)$, that is, interior of $\tilde{D}^{+}(p)$ is not empty. It contradicts to (b).

(a)$\Longleftrightarrow$(c). We can prove this result by a similar way.

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