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ON A CLASS OF ANALYTIC FUNCTIONS SATISFYING Re $\{f'(z)\}>\alpha$

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1. Introduction

Let A be the subclass of functions of the form

(1. 1) $$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

which are analytic in the unit disk $U=\{z: |z|<1\}$.

A function $f(z)$ belonging to A is said to be in the class $P(\alpha)$ if and only if it satisfies

(1. 2) $$\mathrm{Re}\{f'(z)\}>\alpha$$

for some α $(0\leqq\alpha<1)$ and for all $z\in U$. It is well-known that $P(\alpha)$ is the subclass of close-to-convex functions of order α.

For functions $f(z)$ given by (1. 1) and $g(z)$ given by

(1. 3) $$g(z)=z+\sum_{n=2}^{\infty}b_nz^n,$$

we denote by $f*g(z)$ the Hadamard product (or convolution) of $f(z)$ and $g(z)$, that is,

(1. 4) $$f*g(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

Let the functions $f(z)$ and $g(z)$ be analytic in the unit disk U. Then the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $w(z)$ analytic in the unit disk U, with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$, such that

(1. 5) $$f(z)=g(w(z))$$

for $z\in U$. We denote this subordination by

(1. 6) $$f(z)\prec g(z).$$

In particular, if $g(z)$ is univalent in U, the subordination (1. 6) is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

Recently, Fukui, Owa, Ogawa and Nunokawa [1] have given the radius of starlikeness of functions $f(z)$ belonging to the class $P(\alpha)$.

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In this paper, we prove some interesting properties of the class $P(\alpha)$.

2. Some properties of the class $P(\alpha)$

In order to prove our results, we have to recall here the following lemmas.

LEMMA 1. ([2]) *Let* $\mu \geqq 0$ *and*

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{(n+1)^{\mu}}z^{n+1} \qquad (z\in U). \tag{2.1}$$

Then $f(z)$ *is convex in* U.

LEMMA 2. ([3], [4]) *Let* $F(z)$ *and* $G(z)$ *be convex in* U *and* $f(z)\prec F(z)$. *Then*

$$f*G(z)\prec F*G(z). \tag{2.2}$$

With the aid of the above lemmas, we prove

THEOREM. *If the function* $f(z)$ *defined by* (1. 1) *is in the class* $P(\alpha)$, *then*

$$\frac{f(z)}{z}\prec 2\alpha-1-\frac{2(1-\alpha)}{z}\log(1-z). \tag{2.3}$$

The result is best possible.

Proof. Define the function $G(z)$ by

$$G'(z)=\frac{1+(1-2\alpha)z}{1-z} \tag{2.4}$$

and $G(0)=0$. Then, it follows that

$$\frac{G(z)}{z}=2\alpha-1-\frac{2(1-\alpha)}{z}\log(1-z). \tag{2.5}$$

Noting $f(z)\in P(\alpha)$, we see that

$$f'(z)\prec G'(z). \tag{2.6}$$

Defining the function $k(z)$ by

$$k(z)=-\frac{\log(1-z)}{z}=\sum_{n=0}^{\infty}\frac{1}{n+1}z^{n}, \tag{2.7}$$

we have

$$\frac{f(z)}{z}=k*f'(z) \quad\text{and}\quad \frac{G(z)}{z}=k*G'(z). \tag{2.8}$$

Further, $k(z)$ is convex and univalent in U by Lemma 1, and $G'(z)$ is also convex and univalent in U. Therefore, using Lemma 2, we have

$$k*f'(z)\prec k*G'(z), \tag{2.9}$$

that is,

$$\frac{f(z)}{z}\prec\frac{G(z)}{z}=2\alpha-1-\frac{2(1-\alpha)}{z}\log(1-z). \tag{2.10}$$

Thus we complete the proof of Theorem.

Corollary 1. *If the function $f(z)$ defined by* (1.1) *is in the class $P(\alpha)$, then*

$$\text{Re}\left\{\frac{f(z)}{z}\right\}>2\alpha-1+2(1-\alpha)\log 2. \tag{2.11}$$

Proof. Since the function $k(z)$ defined by (2.7) is convex and univalent in U, the function $G_1(z)$ given by

$$G_1(z)=2\alpha-1-\frac{2(1-\alpha)}{z}\log(1-z) \tag{2.12}$$

is also convex and univalent in U. Therefore, by the principle of the subordination, we have

$$\text{Re}\left\{\frac{f(z)}{z}\right\}>\inf_{|z|<1}\text{Re}\{G_1(z)\}. \tag{2.13}$$

We note that $G_1(U)$ is symmetric with respect to the real axis because all coefficients of $G_1(z)$ are real. Noting that $G_1(U)$ is convex, we obtain

$$\inf_{|z|<1}\text{Re}\{G_1(z)\}=\inf_{-1<x<1}G_1(x)=2\alpha-1+2(1-\alpha)\log 2, \tag{2.14}$$

which proves the assertion of Corollary 1.

Corollary 2. *If the function $f(z)$ defined by* (1.1) *is in the class $P(\alpha)$, then*

$$\text{Re}\sqrt{\frac{f(z)}{z}}>\sqrt{2\alpha-1+2(1-\alpha)\log 2}. \tag{2.15}$$

The result is best possible.

Proof. Letting $w=re^{i\theta}$ with $-\pi<\arg(w)<\pi$, we see that

$$\begin{aligned}\text{Re}\sqrt{w}&=\text{Re}(\sqrt{r}\,e^{i(\theta/2)})=\sqrt{r}\cos\frac{\theta}{2}\\&\geqq\sqrt{r|\cos\theta|}=\sqrt{|\text{Re}\{w\}|}.\end{aligned} \tag{2.16}$$

Therefore, using this fact, we get the desired result from Corollary 1. Further, taking the function $G(z)$ defined by

$$G(z)=zG_1(z)\in P(\alpha) \tag{2.17}$$

for $G_1(z)$ given by (2.12), we see that the result is best possible.

REMARK. It follows from Corollary 2 that

$$\text{Re}\sqrt{\frac{f(z)}{z}} \geqq 0.6215\sqrt{1+1.5886\alpha} > 0.6215+0.3784\alpha \tag{2.18}$$

for $f(z)$ belonging to the class $P(\alpha)$.

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