Bull. Korean Math. Soc. 25(1988), No. 2, pp. 215-219

ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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1. Introduction

Let $A(n)$ be the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k \quad (n\in N=\{1, 2, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disk $U=\{z : |z|<1\}$.

A function $f(z)$ belonging to $A(n)$ is said to be in the class $S^*(n, \alpha)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha, \tag{1.2}$$

for some $\alpha(0\leqq\alpha<1)$ and for all $z\in U$.

A function $f(z)$ belonging to $A(n)$ is said to be in the class $B_1(\alpha)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)f^{\alpha-1}(z)}{z^{\alpha}}\right)>0, \tag{1.3}$$

for $\alpha>0$ and for all $z\in U$. (Powers in (1.3) are understood as principal values.) The class $B_1(\alpha)$ is the subclass of Bazilevič functions [5]. It is well known that classes of Bazilevič functions belong to the class of univalent functions.

In this paper, some estimates in relation to the real part of the function $\frac{f(z)}{z}$ are given, where $f(z)$ belong to the class $S^*(n, \alpha)$, $\frac{1}{2}\leqq\alpha<1$, or to the classes $B_1(\alpha)$ for $\alpha=1$ and $\alpha=2$.

2. Some results

We begin with the statement of the following lemma due to Miller and Mocanu [3].

Lemma. *Let $\phi(u, v)$ be a complex valued function,*

$\phi : D\longrightarrow C$, $D\subset C\times C$ (*C is complex plane*),

Received January 13, 1988.

and let $u=u_1+iu_2$ *and* $v=v_1+iv_2$. *Suppose that the function* $\phi(u, v)$ *satisfies the following conditions*:

(i) $\phi(u, v)$ *is continuous in* D;

(ii) $(1, 0)\in D$ *and* $Re\{\phi(1, 0)\}>0$;

(iii) $Re\{\phi(iu_2, v_1)\}\leqq 0$ *for all* $(iu_2, v_1)\in D$ *with*

$$v_1\leqq -n(1+u_2^2)/2.$$

Let $p(z)=1+p_nz^n+p_{n+1}z^{n+1}+\dots$ *be regular in the open unit disk* U *such that* $(p(z), zp'(z))\in D$ *for all* $z\in U$. *If*

$$Re\{\phi(p(z), zp'(z))\}>0 \quad (z\in U),$$

then $Re\{p(z)\}>0$ $(z\in U)$.

Now we prove

THEOREM 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $S^*(n, \alpha)$ *with* $1/2\leqq\alpha<1$. *Then*

$$Re\left(\frac{f(z)}{z}\right)>\frac{n}{2(1-\alpha)+n}. \tag{2.1}$$

Proof. We define the function $p(z)$ by

$$\frac{f(z)}{z}=\beta+(1-\beta)p(z) \tag{2.2}$$

where

$$\beta=\frac{n}{2(1-\alpha)+n}. \tag{2.3}$$

Then the function $p(z)=1+p_nz^n+p_{n+1}z^{n+1}+\dots$ is regular in U. From (2.2) we can express $f(z)$ and then by differentiating we have

$$f'(z)=\beta+(1-\beta)(p(z)+zp'(z)). \tag{2.4}$$

Now, from (2.2) and (2.3) we conclude that

$$\frac{zf'(z)}{f(z)}-\alpha=1-\alpha+\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}. \tag{2.5}$$

Since the function $f(z)$ is in the class $S^*(n, \alpha)$ if and only if $\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha$, we have

$$\mathrm{Re}\left\{1-\alpha+\frac{(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}\right\}>0. \tag{2.6}$$

Letting $p(z)=u=u_1+iu_2$ and $zp'(z)=v=v_1+iv_2$, we define the function $\phi(u, v)$ by

$$\phi(u, v)=1-\alpha+\frac{(1-\beta)v}{\beta+(1-\beta)u}. \tag{2.7}$$

Then $\phi(u, v)$ is continuous in $D=\left(C-\left\{\frac{\beta}{\beta-1}\right\}\right)\times C$ and $\text{Re}\{\phi(1, 0)\}=1-\alpha>0$, and for all $(iu_2, v_1)\in D$ such that $v_1\leqq -n(1+u_2^2)/2$,

$$\begin{aligned}\text{Re}\{\phi(iu_2, v_1)\}&=1-\alpha+\frac{\beta(1-\beta)v}{\beta^2+(1-\beta)^2u_2^2}\\&\leqq 1-\alpha-\frac{\beta(1-\beta)n(1+u_2^2)}{2\{\beta^2+(1-\beta)^2u_2^2\}}\\&\leqq 0.\end{aligned}$$

Thus the function $\phi(u, v)$ satisfies the conditions in Lemma. This proves that $\text{Re}\{p(z)\}>0$, or

$$\text{Re}\left(\frac{f(z)}{z}\right)>\frac{n}{2(1-\alpha)+n}.$$

This completes the proof of Theorem 1.

REMARK. Letting $n=1$ and $\alpha=1/2$, that is, for the class of starlike functions of order 1/2, we have

$$\text{Re}\left(\frac{f(z)}{z}\right)>1/2,$$

which is the well known result [1], [3].

THEOREM 2. *Let* $f(z)\in A(n)$ *and* $\text{Re}\{f'(z)\}>0$. *Then*

$$\text{Re}\left(\frac{f(z)}{z}\right)>\frac{n}{n+2}\ (z\in U).$$

Proof. We define the function $p(z)$ by

$$\left(\frac{n+2}{2}\right)\frac{f(z)}{z}-\frac{n}{2}=p(z). \tag{2.8}$$

Then the function $p(z)$ is regular in U. From (2.8), we have

$$f'(z)=\frac{n}{n+2}+\frac{2}{n+2}(p(z)+zp'(z)),$$

and since $\text{Re}\{f'(z)\}>0$ $(z\in U)$, it follows that

$$\text{Re}\left\{\frac{n}{n+2}+\frac{2}{n+2}(p(z)+zp'(z)\right\}>0\ (z\in U). \tag{2.9}$$

It is easily shown that for the corresponding function

$$\phi(u, v)=\frac{n}{n+2}+\frac{2}{n+2}(u+v)$$

the conditions (i) and (ii) of Lemma are satisfied and that

$$\begin{aligned}\operatorname{Re}\{\phi(iu_2, v_1)\} &= \frac{n}{n+2}+\frac{2}{n+2}v_1 \\ &\leqq \frac{n}{n+2}-\frac{n(1+u_2^2)}{(n+2)} \\ &\leqq 0,\end{aligned}$$

for all (iu_2, v_1) with $v_1 \leqq -n(1+u_2^2)/2$. Therefore, applying Lemma, we have that $\operatorname{Re}\{p(z)\}>0$, that is, that

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)>\frac{n}{n+2} \quad (z\in U).$$

THEOREM 3. *Let the function* $f(z)$ *defined by* (1. 1) *be in the class* $B_1(2)$. *Then*

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)\geqq\frac{n}{n+2} \quad (z\in U).$$

Proof. If we take the change as in (2. 8), and follow the simlar way as in the proof of Theorem 2, we have

$$\text{(2. 10)} \quad \frac{f(z)f'(z)}{z}=\frac{1}{(n+2)^2}(2p(z)+n))^2+\frac{2}{(n+2)^2}(2p(z)+n)zp'(z).$$

Since $\operatorname{Re}\left(\frac{f(z)f'(z)}{z}\right)>0$ $(z\in U)$, that is, from (2. 10),

$$\text{(2. 11)} \quad \operatorname{Re}\left\{\frac{1}{(n+2)^2}(2p(z)+n)^2+\frac{2}{(n+2)^2}(2p(z)+n)zp'(z)\right\}>0 \quad (z\in U).$$

If we consider the function

$$\phi(u, v)=\frac{1}{(n+2)^2}(2u+n)^2+\frac{2}{(n+2)^2}(2u+n)v,$$

then it is directly checked that $\phi(u, v)$ satisfies the conditions (i) and (ii) of Lemma, and for all (iu_2, v_1) such that $v_1 \leqq -n(1+u_2^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\{\phi(iu_2, v_1)\} &= \frac{1}{(n+2)^2}(-4u_2^2+n^2)+\frac{2n}{(n+2)^2}v_1 \\ &\leqq \frac{1}{(n+2)^2}(-4u_2^2+n^2)-\frac{n^2(1+u_2^2)}{(n+2)^2} \\ &\leqq 0.\end{aligned}$$

Therefore, from Lemma, we have that $\mathrm{Re}\{p(z)\}>0$ $(z\in U)$, which implies, bescause of (2.8),

$$\mathrm{Re}\left(\frac{f(z)}{z}\right)>\frac{n}{n+2}\quad (z\in U).$$

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