A GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS II

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1. Introduction

Let $S_p$ denote the class of functions of the form $f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}$ which are analytic and $p$-valent in the unit disc $U=\{z: |z|<1\}$.

For $-1\leq A \leq B \leq 1$, $0 \leq B \leq 1$ and $0 \leq \alpha < p$, let $P(p, A, B, \alpha)$ be the class of those functions $f(z)$ of $S_p$ for which $\frac{f'(z)}{z^{p-1}}$ is subordinate to $p + \frac{pB + (A-B)(p-\alpha)}{1+Bz}$. In other words $f(z) \in P(p, A, B, \alpha)$ if and only if there exists a function $w(z)$ regular in $U$ and satisfying $w(0)=0$, $|w(z)|<1$ for $z \in U$, such that

$$\frac{f'(z)}{z^{p-1}} - p + \frac{pB + (A-B)(p-\alpha)}{1+Bw(z)}w(z).$$

Above condition is equivalent to

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \left| B \frac{f'(z)}{z^{p-1}} - [pB + (A-B)(p-\alpha)] \right| < 1, \quad z \in U. \quad (1.1)$$

Let $T_p$ denote the subclass of $S_p$ consisting of functions analytic and $p$-valent which can be expressed in the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}.$$

We denote by $P^*(p, A, B, \alpha)$ the class obtained by taking intersection of the class $P(p, A, B, \alpha)$ with $T_p$.

The subclasses $T_p^*(A, B, \alpha)$ and $C_p(A, B, \alpha)$ of $T_p$ obtained by replacing $\frac{f'(z)}{z^{p-1}}$ with $\frac{zf'(z)}{f(z)}$ and $\left\{1 + \frac{zf''(z)}{f'(z)} \right\}$ respectively in (1.1) have been studied by the author [1].

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In the present paper, we obtain sharp results concerning coefficient estimates, distortion theorem and radius of convexity for the class $P^*( p, A, B, \alpha )$. It is further shown that the class $P^*( p, A, B, \alpha )$ is closed under "arithmetic mean" and "convex linear combinations".

We also obtain class preserving integral operators of the form

$$ F(z) = \frac{p + c}{z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -p $$

for the class $P^*( p, A, B, \alpha )$. Conversely when $F(z) \in P^*( p, A, B, \alpha )$, radius of $p$-valence of $f(z)$ has been determined. Also we obtain distortion theorem for the fractional integral.

2. Coefficient estimates

**Theorem 1.** A function $f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ is in $P^*( p, A, B, \alpha )$ if and only if

$$ \sum_{n=1}^{\infty} (n+p)(1+B) |a_{n+p}| \leq (B-A)(p-\alpha). $$

The result is sharp.

**Proof.** Let $|z| = 1$. Then

$$ |f'(z) - px^{p-1}| - |Bf'(z) - [pB + (A-B)(p-\alpha)]x^{p-1}| $$

$$ = \left| - \sum_{n=1}^{\infty} (n+p) |a_{n+p}| z^{n+p-1} \right| - (B-A)(p-\alpha) x^{p-1} $$

$$ \leq \sum_{n=1}^{\infty} (n+p)(1+B) |a_{n+p}| - (B-A)(p-\alpha) \leq 0, \text{ by hypothesis.} $$

Hence by the maximum modulus theorem, $f(z) \in P^*( p, A, B, \alpha )$.

Conversely, suppose that

$$ \frac{|f'(z) - px^{p-1}|}{|Bf'(z) - [pB + (A-B)(p-\alpha)]x^{p-1}|} $$

$$ \leq \left| \frac{- \sum_{n=1}^{\infty} (n+p) |a_{n+p}| z^{n+p-1}}{(B-A)(p-\alpha) x^{p-1} - B \sum_{n=1}^{\infty} (n+p) |a_{n+p}| z^{n+p-1}} \right| = 1, \quad z \in U. $$

Since $|\text{Re}(z)| \leq |z|$ for all $z$, we have

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\[
\text{Re} \left\{ \frac{\sum_{n=1}^{\infty} (n+\rho) |a_{n+\rho}| z^{n+\rho-1}}{(B-A)(\rho-\alpha) z^\rho - B \sum_{n=1}^{\infty} (n+\rho) |a_{n+\rho}| z^{n+\rho-1}} \right\} < 1. \tag{2.1}
\]

Choose values of \( z \) on the real axis so that \( f'(z)/z^\rho \) is real. Upon clearing the denominator in (2.1) and letting \( z \to 1 \) through real values, we have

\[
\sum_{n=1}^{\infty} (n+\rho) (1+B) |a_{n+\rho}| \leq (B-A) (\rho-\alpha).
\]

This completes the proof of the theorem.

The function

\[
f(z) = z^\rho - \frac{(B-A)(\rho-\alpha)}{(n+\rho)(1+B)} z^{n+\rho} \quad (n \geq 1)
\]

is an extremal function.

3. Distortion theorem

**Theorem 2.** If \( f(z) \in P^*(p, A, B, \alpha) \), then

\[
r^\rho - \frac{(B-A)(\rho-\alpha)}{(1+p)(1+B)} r^{p+1} \leq |f(z)| \leq r^\rho + \frac{(B-A)(\rho-\alpha)}{(1+p)(1+B)} r^{p+1}(|z|=r),
\]

and

\[
p r^{p-1} - \frac{(B-A)(\rho-\alpha)}{(1+B)} r^\rho \leq |f'(z)| \leq p r^{p-1} + \frac{(B-A)(\rho-\alpha)}{(1+B)} r^\rho (|z|=r).
\]

The estimates are sharp.

**Proof.** From Theorem 1, we have

\[
(1+p)(1+B) \sum_{n=1}^{\infty} |a_{n+\rho}| \leq \sum_{n=1}^{\infty} (n+\rho) (1+B) |a_{n+\rho}| \leq (B-A)(\rho-\alpha).
\]

This implies that

\[
\sum_{n=1}^{\infty} |a_{n+\rho}| \leq \frac{(B-A)(\rho-\alpha)}{(1+p)(1+B)}.
\]

Hence

\[
|f(z)| \leq |z|^\rho + \sum_{n=1}^{\infty} |a_{n+\rho}| |z|^{n+\rho} \leq r^\rho (1+r \sum_{n=1}^{\infty} |a_{n+\rho}|) \\
\leq r^\rho + \frac{(B-A)(\rho-\alpha)}{(1+p)(1+B)} r^{p+1}
\]

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and
\[ |f(z)| \geq |z|^p - \sum_{n=1}^{\infty} |a_{n+p}| |z|^{n+p} \geq r^p (1 - r \sum_{n=1}^{\infty} |a_{n+p}|) \]
\[ \geq r^p \frac{(B-A)(p-\alpha)}{(1+p)(1+B)} r^{p+1}. \]
Thus (3.1) follows.
Also
\[ |f'(z)| \leq |p| |z|^{p-1} + \sum_{n=1}^{\infty} (n+p) |a_{n+p}| |z|^{n+p-1} \]
\[ \leq r^{p-1} (p + r \sum_{n=1}^{\infty} (n+p) |a_{n+p}|) \]
\[ \leq pr^{p-1} \frac{(B-A)(p-\alpha)}{(1+B)} r^p. \]
Similarly
\[ |f''(z)| \geq |p| |z|^{p-1} - \sum_{n=1}^{\infty} (n+p) |a_{n+p}| |z|^{n+p-1} \]
\[ \geq r^{p-1} (p - r \sum_{n=1}^{\infty} (n+p) |a_{n+p}|) \]
\[ \geq pr^{p-1} \frac{(B-A)(p-\alpha)}{(1+B)} r^p. \]
This completes the proof of the theorem.

The bounds are sharp since the equalities are attained for the function
\[ f(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+p)(1+B)} z^{p+1}. \quad (z = \pm r). \]

**Corollary 1.** Let \( f(z) \in P^*(p, A, B, \alpha) \). Then the disc \(|z| < 1\) is mapped onto a domain that contains the disc
\[ |w| < \frac{1+B+p(1+A)+\alpha(B-A)}{(1+p)(1+B)}. \]
The result is sharp for the extremal function
\[ f(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+p)(1+B)} z^{p+1}. \]

Proof follows upon letting \( r \to 1 \) in left hand side of (3.1).

**4. Integral operators**

**Theorem 3.** Let \( c \) be a real number such that \( c > -p \). If \( f(z) \in P^*(p, A, B, \alpha) \), then the function \( F(z) \) defined by
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\[ F(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t) \, dt \]  \hspace{1cm} (4.1)

also belongs to \( P^*(p,A,B,\alpha) \).

**Proof.** Let

\[ f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}. \]

Then from the representation of \( F(z) \), it follows that

\[ F(z) = z^p - \sum_{n=1}^{\infty} |b_{n+p}| z^{n+p}, \]

where

\[ |b_{n+p}| = \frac{(p+c)}{(n+p+c)} |a_{n+p}|. \]

Therefore

\[ \sum_{n=1}^{\infty} (n+p)(1+B) |b_{n+p}| = \sum_{n=1}^{\infty} (n+p)(1+B) \frac{p+c}{(n+p+c)} |a_{n+p}| \]

\[ \leq \sum_{n=1}^{\infty} (n+p)(1+B) |a_{n+p}| \]

\[ \leq (B-A)(p-\alpha), \]

since \( f(z) \in P^*(p,A,B,\alpha) \). Hence, by Theorem 1,

\[ F(z) \in P^*(p,A,B,\alpha). \]

**Theorem 4.** Let \( c \) be a real number, \( c \geq p \). If \( F(z) \in P^*(p,A,B,\alpha) \), then the function \( f(z) \) defined in (4.1) is \( p \)-valent for \( |z| < R_\rho^* \), where

\[ R_\rho^* = \inf_{n \geq 1} \left[ \left( \frac{p+c}{n+p+c} \right) \cdot \frac{(1+B)p}{(B-A)(p-\alpha)} \right]^{1/2}. \]

The result is sharp.

**Proof.** Let

\[ F(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p}. \]

It follows then from (4.1) that

\[ f(z) = \frac{z^{1-c}}{p+c} \frac{d}{dz} (z^c F(z)) \]

\[ = z^p - \sum_{n=1}^{\infty} \left( \frac{n+p+c}{p+c} \right) |a_{n+p}| z^{n+p}. \]

To prove the result it suffices to show that
\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| < p \quad \text{for} \quad |z| < R_p^*.
\]

Now
\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| -\sum_{n=1}^{\infty} (n+p) \left( \frac{n+p+c}{p+c} \right) a_{n+p} z^n \right|
\leq \sum_{n=1}^{\infty} (n+p) \left( \frac{n+p+c}{p+c} \right) |a_{n+p}| |z|^n.
\]

Thus
\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| < p \quad \text{if}
\sum_{n=1}^{\infty} (n+p) \left( \frac{n+p+c}{p+c} \right) |a_{n+p}| |z|^n < p. \tag{4.2}
\]

But Theorem 1 confirms that
\[
\sum_{n=1}^{\infty} \frac{(n+p)(1+B)p}{(B-A)(p-\alpha)} |a_{n+p}| \leq p.
\]

Thus (4.2) will be satisfied if
\[
(n+p) \left( \frac{n+p+c}{p+c} \right) |a_{n+p}| |z|^n \leq \frac{(n+p)(1+B)p}{(B-A)(p-\alpha)} |a_{n+p}|, \quad n=1, 2, \ldots
\]
or if
\[
|z| \leq \left[ \left( \frac{p+c}{n+p+c} \right) \frac{(1+B)p}{(B-A)(p-\alpha)} \right]^\frac{1}{n}. \tag{4.3}
\]

The required result follows now from (4.3).

The result is sharp for the function
\[
f(z) = z^p - \frac{(n+p+c)(B-A)(p-\alpha)}{(n+p)(p+c)(1+B)} z^{n+p}.
\]

5. Radius of convexity for the class \( P^*(p, A, B, \alpha) \)

**Theorem 5.** If \( f(z) \in P^*(p, A, B, \alpha) \), then \( f(z) \) is \( p \)-valently convex in the disc
\[
|z| < R_p = \inf_n \left[ \frac{(1+B)p^2}{(B-A)(p-\alpha)(n+p)} \right]^\frac{1}{n}, \quad n=1, 2, 3, \ldots.
\]
The result is sharp.

**Proof.** To prove the theorem it is sufficient to show that
\[
\left| \left( 1 + \frac{zf''(z)}{f'(z)} \right) - p \right| \leq p \quad \text{for} \quad |z| < R_p.
\]

We have
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\[
\left| 1 + \frac{zf''(z)}{f'(z)} \right| - p = \left| \frac{\sum_{n=1}^{\infty} n(n+p) |a_{n+p}| z^n}{p - \sum_{n=1}^{\infty} n(n+p) |a_{n+p}| z^n} \right| \leq \frac{\sum_{n=1}^{\infty} n(n+p) |a_{n+p}| |z|^n}{p - \sum_{n=1}^{\infty} (n+p) |a_{n+p}| |z|^n}.
\]

Thus

\[
\left| 1 + \frac{zf''(z)}{f'(z)} \right| - p \leq p \quad \text{if}
\]

\[
\frac{\sum_{n=1}^{\infty} n(n+p) |a_{n+p}| |z|^n}{p - \sum_{n=1}^{\infty} (n+p) |a_{n+p}| |z|^n} \leq p
\]

or

\[
\sum_{n=1}^{\infty} \left( \frac{n+p}{p} \right)^2 |a_{n+p}| |z|^n \leq 1.
\]

But from Theorem 1, we obtain

\[
\sum_{n=1}^{\infty} \frac{(n+p)(1+B)}{(B-A)(p-\alpha)} |a_{n+p}| \leq 1.
\]

Hence \( f(z) \) is \( p \)-valently convex if

\[
\left( \frac{n+p}{p} \right)^2 |z|^n \leq \frac{(n+p)(1+B)}{(B-A)(p-\alpha)}
\]

or

\[
|z| \leq \left[ \frac{(1+B)p^2}{(B-A)(p-\alpha)(n+p)} \right]^{\frac{1}{n}}, \quad n=1, 2, 3, \ldots.
\]

This completes the proof of the theorem.

The result is sharp for the function

\[
f(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} z^{n+p}.
\]

6. Closure theorems

Theorem 6. If \( f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}| z^{n+p} \) and \( g(z) = z^p - \sum_{n=1}^{\infty} |b_{n+p}| z^{n+p} \) are in \( P^*(p, A, B, \alpha) \), then the function \( h(z) = z^p - \frac{1}{2} \sum_{n=1}^{\infty} |a_{n+p} + b_{n+p}| z^{n+p} \)
is also in \( P^*(p, A, B, \alpha) \).

**Proof.** Since \( f(z) \) and \( g(z) \) are in \( P^*(p, A, B, \alpha) \). Therefore we have

\[
\sum_{n=1}^{\infty} (n+p) (1+B) |a_{n+p}| \leq (B-A) (p-\alpha) \tag{6.1}
\]

and

\[
\sum_{n=1}^{\infty} (n+p) (1+B) |b_{n+p}| \leq (B-A) (p-\alpha). \tag{6.2}
\]

From (6.1) and (6.2) we obtain

\[
\frac{1}{2} \sum_{n=1}^{\infty} (n+p) (1+B) |a_{n+p} + b_{n+p}| \leq (B-A) (p-\alpha).
\]

This completes the proof of the theorem.

**Theorem 7.** Let \( f_i(z) = z^p - \sum_{n=1}^{\infty} |a_{i,n+p}| z^{n+p} \) be in the classes \( P^*(p, A, B, \alpha_i) \) for each \( i = 1, 2, 3, \ldots, m \). Then the function

\[
h(z) = z^p - \frac{1}{m} \sum_{n=1}^{m} \left( \sum_{i=1}^{m} |a_{i,n+p}| \right) z^{n+p}
\]

is in the class \( P^*(p, A, B, \alpha) \), where \( \alpha = \min \{ \alpha_i \}_{1 \leq i \leq m} \).

**Proof.** Since \( f_i(z) \in P^*(p, A, B, \alpha_i) \) for each \( i = 1, 2, \ldots, m \), we have

\[
\sum_{n=1}^{\infty} (n+p) (1+B) |a_{i,n+p}| \leq (B-A) (p-\alpha_i)
\]

by Theorem 1. Hence we obtain

\[
\sum_{n=1}^{\infty} (n+p) \left( \frac{1}{m} \sum_{i=1}^{m} |a_{i,n+p}| \right)
\]

\[
= \frac{1}{m} \sum_{n=1}^{\infty} \left( \sum_{i=1}^{m} (n+p) |a_{i,n+p}| \right)
\]

\[
\leq \frac{1}{m} \sum_{i=1}^{m} \left( \frac{(B-A) (p-\alpha_i)}{(1+B)} \right)
\]

\[
\leq \frac{(B-A) (p-\alpha)}{(1+B)},
\]

because \( \frac{(B-A) (p-\alpha')}{(1+B)} \geq \frac{(B-A) (p-\alpha'')}{(1+B)} \) for \( \alpha' \leq \alpha'' \).

Thus we get

\[
\sum_{n=1}^{\infty} (n+p) (1+B) \left( \frac{1}{m} \sum_{i=1}^{m} |a_{i,n+p}| \right) \leq (B-A) (p-\alpha)
\]

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which shows that \( h(z) \in P^*(p, A, B, \alpha) \).

**Theorem 8.** Let \( f_p(z) = z^p, f_{n+p}(z) = z^p - \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} z^{n+p}, n=1, 2, 3, \ldots \). Then \( f(z) \in P^*(p, A, B, \alpha) \) if and only if it can be expressed in the form \( f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) \) where \( \lambda_{n+p} \geq 0 \) and \( \sum_{n=0}^{\infty} \lambda_{n+p} = 1 \).

**Proof.** Suppose \( f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z) \)

\[
= z^p - \sum_{n=1}^{\infty} \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} \lambda_{n+p} z^{n+p}.
\]

Then

\[
\sum_{n=1}^{\infty} \frac{(1+B)(n+p)}{(B-A)(p-\alpha)} \lambda_{n+p} \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} = \sum_{n=1}^{\infty} \lambda_{n+p} = 1 - \lambda_p \leq 1.
\]

Hence by Theorem 1, \( f(z) \in P^*(p, A, B, \alpha) \).

Conversely, suppose that \( f(z) \in P^*(p, A, B, \alpha) \). Since

\[
|a_{n+p}| \leq \frac{(B-A)(p-\alpha)}{(1+B)(n+p)} \quad (n=1, 2, 3, \ldots),
\]

we may set

\[
\lambda_{n+p} = \frac{(1+B)(n+p)}{(B-A)(p-\alpha)} |a_{n+p}| \quad (n=1, 2, 3, \ldots)
\]

and

\[
\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_{n+p}.
\]

Then

\[
f(z) = \sum_{n=0}^{\infty} \lambda_{n+p} f_{n+p}(z).
\]

This completes the proof of the theorem.

**Remarks.**

(1) Putting \( \alpha = 0 \) in the above theorems we get the results obtained by Shukla and Dashrath [4].

(2) Putting \( p = 1 \) and taking \( A = -\beta, B = \beta, 0 < \beta \leq 1 \) in the above theorems we get the results obtained by Gupta and Jain [2].
7. Fractional integral

In 1978, Owa [3] gave the following definition for the fractional integral.

**Definition 1.** The fractional integral of order \( k \) is defined by
\[
D_z^{-k}f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\zeta) \, d\zeta}{(z-\zeta)^{1-k}},
\]
where \( k > 0 \), \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin and the multiplicity of \((z-\zeta)^{k-1}\) is removed by requiring \( \log (z-\zeta) \) to be real when \((z-\zeta)>0\).

**Theorem 9.** Let a function \( f(z) = z^{\rho} - \sum\limits_{n=1}^{\infty} a_{n+\rho} \, z^{n+\rho} \) be in the class \( P^*(\rho, A, B, \alpha) \). Then we have
\[
|D_z^{-k}f(z)| \geq \frac{\Gamma(1+\rho)}{\Gamma(1+\rho+k)} \, |z|^{\rho+k} \left[ 1 - \frac{1}{1+\rho+k} \cdot \frac{(B-A)(\rho-\alpha)}{(1+B)} \right] \, |z|
\]
and
\[
|D_z^{-k}f(z)| \leq \frac{\Gamma(1+\rho)}{\Gamma(1+\rho+k)} \, |z|^{\rho+k} \left[ 1 + \frac{1}{1+\rho+k} \cdot \frac{(B-A)(\rho-\alpha)}{(1+B)} \right] \, |z|
\]
for \( 0 < k < 1 \) and \( z \in U \). The result is sharp.

**Proof.** Let
\[
F(z) = \frac{\Gamma(1+\rho+k)}{\Gamma(1+\rho)} \, z^{-k}D_z^{-k}f(z)
\]
\[
= z^{\rho} - \sum\limits_{n=1}^{\infty} \frac{\Gamma(n+\rho+1)\Gamma(1+\rho+k)}{\Gamma(n+\rho+1+k)\Gamma(1+\rho)} \, |a_{n+\rho}| \, z^{n+\rho}
\]
\[
= z^{\rho} - \sum\limits_{n=1}^{\infty} A(n) \, |a_{n+\rho}| \, z^{n+\rho},
\]
where
\[
A(n) = \frac{\Gamma(n+\rho+1)\Gamma(1+\rho+k)}{\Gamma(n+\rho+1+k)\Gamma(1+\rho)} \quad (n \geq 1).
\]
Since
\[
0 < A(n) \leq A(1) = \frac{1+\rho}{1+\rho+k},
\]
we have, with the help of Theorem 1,
\[
|F(z)| \geq |z|^{\rho} - A(1) \, |z|^{1+\rho} \sum\limits_{n=1}^{\infty} |a_{n+\rho}|
\]
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\[ \geq |z|^\rho - \frac{1+\rho}{1+\rho+k} \cdot \frac{(B-A)(\rho-\alpha)}{(1+\rho)(1+B)} |z|^{1+\rho} \]

\[ \geq |z|^\rho - \frac{1}{1+\rho+k} \cdot \frac{(B-A)(\rho-\alpha)}{(1+B)} |z|^{1+\rho} \]

and

\[ |F(z)| \leq |z|^\rho + A(1)|z|^{1+\rho} \sum_{n=1}^{\infty} |a_{n+\rho}| \]

\[ \leq |z|^\rho + \frac{1}{1+\rho+k} \cdot \frac{(B-A)(\rho-\alpha)}{(1+B)} |z|^{1+\rho} \]

which prove the inequalities of Theorem 9. Further, equalities are attained for the function

\[ D_z^{-k}f(z) = \frac{\Gamma(1+\rho)}{\Gamma(1+\rho+k)} z^{\rho+k} \left\{ 1 - \frac{1}{1+\rho+k} \cdot \frac{(B-A)(\rho-\alpha)}{(1+B)} z \right\}, \]

or

\[ f(z) = z^{\rho} - \frac{(B-A)(\rho-\alpha)}{(1+\rho)(1+B)} z^{1+\rho}. \]

**Corollary 2.** Under the hypotheses of Theorem 9, \( D_z^{-k}f(z) \) is included in the disc with center at the origin and radius

\[ \frac{\Gamma(1+\rho)}{\Gamma(1+\rho+k)} \left\{ 1 + \frac{1}{1+\rho+k} \cdot \frac{(B-A)(\rho-\alpha)}{(1+B)} \right\}. \]

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