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ITERATIONS OF THE UNIT SINGULAR INNER FUNCTION

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1. Introduction

Let $M(z)=\exp\left(-\frac{1+z}{1-z}\right)$ be the unit singular inner function. See [1] or [2] for the basic facts about inner functions. We define the iterations of $M(z)$ as

$$M_{n+1}(z)=M_n\circ M(z),\quad M_1(z)=M(z),\quad n=1,2,\dots.$$

Since the composition $M_2(z)=M\circ M(z)$ is known (see [5] for example) to be a singular inner function it has the "cannonical" representation

$$M_2(z)=e^{i\gamma}\exp\left(-\int_T\frac{\xi+z}{\xi-z}d\mu(\xi)\right),\quad \gamma \text{ real},$$

where μ is a finite, positive singular Borel measure on the unit circle T. In section 2, we have explicit cannonical representation of $M_2(z)$ by determining the singular measure μ. In section 3 we show that

$$\lim_{n\to\infty} M_n(z)=0.213652452\cdots,\quad |z|<1.$$

These facts might have been known but could not be found in the literature.

2. $M\circ M(z)$

We need the following fact.

PROPOSITION A[3]. *If ϕ is a singular inner function and σ is its associated singular measure, then the mass of σ at $\xi\in T$ is given by*

$$\sigma(\{\xi\})=-\lim_{r\to 1}\frac{1-r}{2}\log|\phi(r\xi)|.$$

Since $M_2(z)=\exp\left(-\frac{1+M(z)}{1-M(z)}\right)$, the associated singular measure μ would carry point masses at the zeros of $M(z)-1=0$, which are

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$$z=\xi_n=\frac{2n\pi i-1}{2n\pi i+1},\quad n \text{ integers.}$$

To compute the magnitude of the mass of μ at each ξ_n we need some computational lemmas.

LEMMA 1. $1-|M(r\xi_n)|^2\sim\dfrac{2(1-r^2)(4n^2\pi^2+1)}{(1+r)^2+4n^2\pi^2(1-r)^2}$ *as* $r\to1$, *where* $A(r)\sim B(r)$ *as* $r\to1$ *means* $A(r)/B(r)\to1$ *as* $r\to1$.

Proof.

$$\begin{aligned}1-|M(r\xi_n)|^2&=1-\exp\left(-\frac{2(1-r^2)}{|1-r\xi_n|^2}\right)\\&=\frac{2(1-r^2)}{|1-r\xi_n|^2}-\frac{1}{2!}\left(\frac{2(1-r^2)}{|1-r\xi_n|^2}\right)^2+\cdots\\&\sim\frac{2(1-r^2)}{|1-r\xi_n|^2}\quad\text{as } r\to1.\end{aligned}$$

Since we easily compute

$$|1-r\xi_n|^2=\frac{(1+r)^2+4n^2\pi^2(1-r)^2}{4n^2\pi^2+1},$$

the lemma follows.

LEMMA 2. $|1-M(r\xi_n)|^2\sim\dfrac{(1-r)^2(4n^2\pi^2+1)^2}{(1+r)^2+4n^2\pi^2(1-r)^2}$ *as* $r\to1$.

Proof.

$$\begin{aligned}1-M(r\xi_n)&=1-\exp\left(-\frac{(1-r^2)+r(\xi_n-\bar{\xi}_n)}{|1-r\xi_n|^2}\right)\\&=1-\exp\left(-\frac{(1-r^2)(4n^2\pi^2+1)+8rn\pi i}{(1+r)^2+4n^2\pi^2(1-r)^2}+2n\pi i\right)\\&=1-\exp\left(-\frac{(1-r^2)(4n^2\pi^2+1)-2n\pi i(1-r)^2(4n^2\pi^2+1)}{(1+r)^2+4n^2\pi^2(1-r)^2}\right)\\&\sim\frac{(1-r)(4n^2\pi^2+1)[(1+r)-2n\pi(1-r)i]}{(1+r)^2+4n^2\pi^2(1-r)^2}.\end{aligned}$$

Therefore, we have

$$|1-M(r\xi_n)|^2\sim\frac{(1-r)^2(4n^2\pi+1)^2}{(1+r)^2+4n^2\pi^2(1-r)^2}.$$

We now prove our main

THEOREM 3. $M\circ M(z)=\exp\left(-\sum_{-\infty}^{\infty}\dfrac{\xi_n+z}{\xi_n-z}\ \dfrac{2}{4n^2\pi^2+1}\right).$ (1)

That is, $\mu=\sum_{-\infty}^{\infty}\frac{2}{4n^2\pi^2+1}\delta_n$, *where* δ_n *is the unit mass at the point* ξ_n. *Incidently, we have*

$$\frac{1+M(z)}{1-M(z)}=\sum_{-\infty}^{\infty}\frac{\xi_n+z}{\xi_n-z}\,\frac{2}{4n^2\pi^2+1}.$$

Proof. We apply Proposition A and use Lemmas 1 and 2 to compute the mass of μ at ξ_n as

$$\begin{aligned}\mu(\{\xi_n\})&=\lim_{r\to1}\frac{1-r}{2}\,\frac{1-|M(r\xi_n)|^2}{|1-M(r\xi_n)|^2}\\&=\lim_{r\to1}\frac{1-r}{2}\,\frac{2(1-r^2)(4n^2\pi^2+1)}{(1+r)^2+4n^2\pi^2(1-r)^2}\cdot\frac{(1+r)^2+4n^2\pi^2(1-r)^2}{(1-r)^2(4n^2\pi^2+1)^2}\\&=\frac{2}{4n^2\pi^2+1}.\end{aligned}$$

If we note that

$$\begin{aligned}\mu(T)&=-\log|M\circ M(0)|=\frac{1+e^{-1}}{1-e^{-1}}\\&=\sum_{-\infty}^{\infty}\frac{2}{4n^2\pi^2+1},\quad \text{(See [4, p. 50])}\end{aligned}$$

we see that $\mu=\sum_{-\infty}^{\infty}\frac{2}{4n^2\pi^2+1}\delta_n$. Comparing the values of the both sides of (1) at $z=0$, we have the identity (1). This completes the proof.

3. $\lim M_n$

For the real values x $(-1<x<1)$, $M(x)=\exp\left(-\frac{1+x}{1-x}\right)$ is decreasing from 1 to 0 as x varies from -1 to 1. Therefore $M(x)$ has a unique fixed point λ on the interval $(-1,1)$ i.e., $M(\lambda)=\lambda$. We compute $\lambda=0.213652452\cdots$ by the Newton's method. We now prove

THEOREM 4. $\lim_{n\to\infty}M_n(z)=\lambda$, $|z|<1$.

Proof. We set $\alpha_j=M_j(0)$, $j=1,2,\cdots$. Since $0<M(0)$ and M is strictly decreasing on the interval $(-1,1)$, we easily see that

$$\alpha_2<\alpha_4<\cdots<\alpha_{2j}<\cdots<\alpha_{2j+1}<\cdots<\alpha_3<\alpha_1,\quad j=1,2,3,\cdots.$$

The sequences $\{\alpha_{2j}\}$ and $\{\alpha_{2j+1}\}$ converge. Let

$$\lim_{j\to\infty}\alpha_{2j}=\varepsilon \text{ and } \lim_{j\to\infty}\alpha_{2j+1}=\delta.$$

We note that for $j=1, 2, 3, \cdots$ we have

$$M_{2n}(\alpha_{2j})=\alpha_{2(n+j)}\to\varepsilon \quad \text{as } n\to\infty$$

and

$$M_{2n+1}(\alpha_{2j})=\alpha_{2(n+j)+1}\to\delta \text{ as } n\to\infty.$$

By the Vitali's convergence theorem [6, p. 168], $\lim_{n\to\infty} M_{2n}(z)$ converges uniformly on the compact subsets of the unit disc $|z|<1$. Therefore $L(z)=\lim_{n\to\infty} M_{2n}(z)$ defines a holomorphic function on $|z|<1$. Since

$$L(\alpha_{2j})=\lim_{n\to\infty} M_{2n}(\alpha_{2j})=\lim_{n\to\infty} \alpha_{2(n+j)}=\varepsilon$$

for all $j=1, 2, \cdots$, we have $L(z)\equiv\varepsilon$ for all $|z|<1$. But $L(\lambda)=\lambda$; so $L(z)\equiv\lambda\equiv\varepsilon$.

Similarily, we have

$$N(z)\equiv\lim_{n\to\infty} M_{2n+1}(z)\equiv\lambda, \quad |z|<1.$$

Therefore we have

$$\lim_{n\to\infty} M_n(z)\equiv\lambda, \quad |z|<1.$$

This completes the proof.

COROLLARY. *λ is the only fixed point of M in the unit disc $|z|<1$.*

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