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PSEUDO-CONJUGATIONS

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This note gives a combinatorial treatment to the problem finding a generating set among conjugating automorphisms of a free group and to the method deciding when a conjugating endomorphism of a free group is an automorphism. Our group of pseudo-conjugating automorphisms can be thought of as a generalization of the Artin's braid group.

One of main results of the note was motivated from link cobordisms and was utilized in [3]. But it seems worthwhile to look in the purely group theoretic aspect of the result. The idea of developing generator-conjugations to pseudo-conjugations was suggested by John McCarthy.

Let F_n be the free group on letters $x_1, \dots, x_n$. This set of generators will be fixed throughout the paper. An endomorphism σ of F_n is a *pseudo-conjugation* if σ maps generators to conjugates of generators, more precisely, for each $1 \le i \le n$

$$\sigma(x_i) = w_{\sigma,i} x_{p\sigma(i)} w_{\sigma,i}^{-1}$$

where $w_{\sigma,i}$ is a word in F_n and $p\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is a function. If $p\sigma$ is the identity, the endomorphism σ is called a *generator-conjugation*. By the properties of free groups, it is easy to see that σ determines p_σ uniquely. On the other hand, $w_{\sigma,i}$ is only determined by σ up to a right multiplication by a power of $x_{p\sigma(i)}$. That is, we may replace $w_{\sigma,i}$, only by $w_{\sigma,i} x_{p\sigma(i)}{}^m$ for some $m \in \boldsymbol{Z}$. σ determines $w_{\sigma,i}$ uniquely if we assume that $w_{\sigma,i} x_{p\sigma(i)} w_{\sigma,i}^{-1}$ is freely reduced. From here on if σ is a pseudo-conjugation then $w_{\sigma,i}$ will be defined by the free-reduction of $\sigma(x_i) = w_{\sigma,i} x_{p\sigma(i)} w_{\sigma,i}^{-1}$.

Let $\mathrm{PC}(F_n)$ denote the monoid of all pseudo-conjugations of F_n. Let T_n be the monoid of maps: $\{1, \dots, n\} \to \{1, \dots, n\}$. Then the map sending σ in $\mathrm{PC}(F_n)$ to $p\sigma$ in T_n is a homomorphism.

Similarly let $\mathrm{PCA}(F_n)$ denote the group of pseudo-conjugating automorphisms of F_n. Let S_n be the group of permutations on $\{1, \dots, n\}$.

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Passing to the abelianizations, one sees that p_σ in S_n is a homomorphism. Let CA(F_n) be the kernel of π and be called the group of *generator-conjugating automorphisms.*

For $p \in S_n$, let $\rho(p)$ defined to be an automorphism sending x_i to $x_{p(i)}$ for $1 \leq i \leq n$. Then ρ is a monomorphism and

$$1 \to \mathrm{CA}(F_n) \to \mathrm{PCA}(F_n) \xrightarrow{\pi} S_n \to 1$$

becomes a split short exact sequence with splitting ρ. Hence for a given $\sigma \in \mathrm{PCA}(F_n)$, there are a unique $p \in S_n$ and a unique $\alpha \in \mathrm{CA}(F_n)$ such that $\sigma = p \circ \alpha$. In fact, α should be given to be $\rho(p\sigma^{-1}) \circ \sigma$.

Let AUF(F_n) denote the group of automorphisms of F_n. We recall that the n-th braid group B_n can be defined as the subgroup of AUT (F_n) that is generated by the automorphisms β_i for $i=1, \dots, n-1$ defined by $\beta_i(x_i) = x_i x_{i+1} x_i^{-1}$, $\beta_i(x_{i+1}) = x_i$ and $\beta_i(x_j) = x_j$ for $j \neq i, i+1$. Over the fixed basis $\{x_1, \dots, x_n\}$, B_n is a subgroup of PCA(F_n).

In 1925, E. Artin characterized the element of B_n ([1], see also [2]). Let β be an endomorphism of F_n. Then $\beta \in B_n$ if an only if β satisfies the two conditions:

$$\beta(x_i) = w_i x_{p(i)} w_i^{-1}, \quad 1 \leq i \leq n;$$
$$\beta(x_1 \cdots x_n) = x_1 \cdots x_n,$$

where $p \in S_n$ and w_i is a word in the letters $x_1, \dots, x_n$. This gives a simple condition which ensures that an endomorphism of F_n is an automorphism of F_n. It is an interesting open problem to find a "simple" necessary and sufficient condition for this. We will present a simple necessary condition for a pseudo-conjugation to be an automorphism.

Now we discuss generators for PCA(F_n). $\sigma \in \mathrm{AUT}(F_n)$ is called a *permutation* if $\sigma = \rho(p)$ for some $p \in S_n$. $\sigma \in \mathrm{AUT}(F_n)$ is called a *simple conjugation* if there exists a pair of distinct integers $i, j = 1, \dots, n$ such that $\sigma = \alpha_{i,j}$ where $\alpha_{i,j}$ satisfies

$$\alpha_{i,j}(x_i) = x_j x_i x_j^{-1};$$
$$\alpha_{i,j}(x_k) = x_k \quad \text{for} \quad k \neq i.$$

THEOREM. *$CA(F_n)$ is generated by simple conjugations.*

In view of the above split short exact sequence, we have:

COROLLARY. *$PCA(F_n)$ is generated by permutations and simple conju-*

gations.

$\sigma \in \mathrm{AUT}(F_n)$ is a *generalized simple braid* if there exists a pair of distinct integers $i, j=1, \ldots, n$ such that $\sigma=\sigma_{i,j}$ where $\sigma_{i,j}$ satisfies

$$\sigma_{i,j}(x_i)=x_i x_j x_i^{-1};$$
$$\sigma_{i,j}(x_j)=x_i;$$
$$\sigma_{i,j}(x_k)=x_k, \quad k \neq i, j.$$

Artin's generators for B_n are just $\sigma_{i,i+1}$ for $1 \leq i \leq n-1$. Let (i, j) be the transposition of i and j in S_n. Then

$$\alpha_{i,j}=(i, j) \circ \sigma_{i,j} \quad \text{for } 1 \leq i \neq j \leq n$$

Therefore we have:

COROLLARY. *$PCA(F_n)$ is generated by permutations and generalized simple braids.*

Suppose α is a generator-conjugation given by

$$\alpha(x_i)=w_i x_i w_i^{-1} \quad \text{for } i=1, \ldots, n$$

where w_i is a word and $w_i x_i w_i^{-1}$ is freely reduced. It is convenient to write $\alpha=(w_1, w_2, \ldots, w_n)$. This expression is unique. The simple conjugation $\alpha_{i,j}$ is now written as $(e, \ldots, e, x_j, e, \ldots, e)$, and $\alpha_{i,j}^{-1}$ is given by $(e, \ldots, e, x_j^{-1}, e, \ldots, e)$ where $x_j^{\pm 1}$ is in the i-th spot and e denotes the empty word.

LEMMA. *Let $\alpha=(w_1, w_2, \ldots, w_n)$ be a generator-conjugation. If α is an automorphism and is not the identity, then there is at least a pair (j, k) of distinct integers among $1, 2, \ldots, n$ such that*

$$w_k=w_j x_j^{\varepsilon} \bar{w}_k$$

where $\bar{w}_k$ is a reduced word which does not end with x_k or x_k^{-1} and $\varepsilon=\pm 1$.

Applying a permutation to this lemma, we have:

COROLLARY. *Let σ be a pseudo-conjugation as given at the beginning of the paper. If σ is an automorphism and is not a permutation, then there is at least a pair (j, k) of distinct integers among $1, 2, \ldots, n$ such that*

$$w_{\sigma,k}=w_{\sigma,j} x_{p\sigma(j)}^{\varepsilon} \bar{w}_k$$

where $\bar{w}_k$ is a reduced word which does not end with $x_{p\sigma(j)}$ or $x_{p\sigma(j)}^{-1}$

and $\varepsilon=\pm 1$.

Proof of Lemma. Let $\alpha^{-1}=(u_1, u_2, \ldots, u_n)$. In fact, $u_i=\alpha^{-1}(w_i)$. Since α is not the identity, there is $1\leq k\leq n$ such that w_k is not the empty word and neither is u_k. If u_k is given by the word $x_{k_1}{}^{\varepsilon_1}x_{k_2}{}^{\varepsilon_2}\cdots x_{k_r}{}^{\varepsilon_r}$, then

$$x_k=\alpha\circ\alpha^{-1}(x_k)=\left(\prod_{s=1}^{r}(w_{k_s}x_{k_s}{}^{\varepsilon_s}w_{k_s}{}^{-1})\right)w_kx_kw_k{}^{-1}\left(\prod_{s=1}^{r}(w_{k_s}x_{k_s}{}^{\varepsilon_s}w_{k_s}{}^{-1})\right)^{-1}.$$

Since $w_ix_iw_i{}^{-1}$ was reduced for each i, so is $w_{k_s}x_{k_s}{}^{\varepsilon_s}w_{k_s}{}^{-1}$ for each s. Thus in the above product any cancelation must begin between one of pairs $(w_{k_r}{}^{-1}, w_k)$ or $(w_{k_s}{}^{-1}, w_{s+1})$ for $s=1, \ldots, r-1$. We now assume that Lemma is false. Then there is no way for any of the middle letters in the block, i. e., x_k and $x_{k_1}{}^{\varepsilon_1}, x_{k_2}{}^{\varepsilon_2}, \cdots, x_{k_r}{}^{\varepsilon_r}$ in the above product, to be canceled out. Since there is at least one letter among these middle letters which is different from x_k or $x_k{}^{-1}$ (for example $x_{k_r}{}^{\varepsilon_r}$), this is a contradiction.

When $\alpha=(w_1, w_2, \ldots, w_n)$ be a generator-conjugation, the *length* $l(\alpha)$ of α is defined as $l(\alpha)=l(w_1)+l(w_2)+\cdots+l(w_n)$ where $l(w_i)$ is the length of the reduced word w_i. It is clear that $l(\alpha)=0$ if and only if α is the identity.

Proof of Theoem. Let $\alpha=(w_1, w_2, \ldots, w_n)$ be in $CA(F_n)$. Our proof is done by induction on $l(\alpha)$. Suppose that α is not the identity. Let (j, k) be the pair given by Lemma. Since w_k is reduced, $\bar{w}_k$ can not start with $x_j{}^{-\varepsilon}$. Then $\alpha\circ\alpha_{k,j}{}^{-\varepsilon}=(w_1, \ldots, w_{k-1}, w_j\bar{w}_k, w_{k+1}, \ldots, w_n)$. Thus $l(\alpha\circ\alpha_{k,j}{}^{-\varepsilon})$ is strictly less than $l(\alpha)$. By induction, α can be made into the identity by composing $\alpha_{k,j}{}^{\varepsilon}$'s on the right.

Lemma and the proof of Theorem give an effective method for determining whether a pseudo-conjugation is an automorphism.

Let F_3 be the free group on letters x, y, z. Define a pseudo-conjugation $\sigma : F_3\to F_3$ by $\sigma(x)=yxyx^{-1}y^{-1}$, $\sigma(y)=x$, $\sigma(z)=z$. After permuting x and y, we have a generator-conjugation (yx, e, e). By Lemma, this is not an automorphism.

For another example, let $\alpha=(z^{-1}y, x, xyx^{-1})$ be a generator-conjugation on F_3. Noticing $w_3=w_2yx^{-1}$, we have $\alpha\circ\alpha_{3,2}{}^{-1}=(z^{-1}y, x, e)$ Applying the same technique again, $\alpha\circ\alpha_{3,2}{}^{-1}\circ\alpha_{1,3}=(y, x, e)$. Now this conjugation does not satisfy the condition of Lemma and hence α was not an automorphism.

Theorem and its corollaries say that every automorphism in $CA(F_n)$ or in $PCA(F_n)$ can be written as a product of $\alpha_{i,j}$'s or a product of $\alpha_{i,j}$'s and a permutation. However the product is by no means unique. For example, $\alpha_{i,j}\circ\alpha_{k,j}=\alpha_{k,j}\circ\alpha_{i,j}$. The problem finding the complete set of relations among these generators of $CA(F_n)$ seems interesting. In [3], the quotient group of $CA(F_n)$ by inner automorphisms was useful and hence the same problem for this group needs some attention. These problems will be considered in sequent papers.

References

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