PSEUDO–CONJUGATIONS

KI HYOUNG KO

This note gives a combinatorial treatment to the problem finding a generating set among conjugating automorphisms of a free group and to the method deciding when a conjugating endomorphism of a free group is an automorphism. Our group of pseudo–conjugating automorphisms can be thought of as a generalization of the Artin’s braid group.

One of main results of the note was motivated from link cobordisms and was utilized in [3]. But it seems worthwhile to look in the purely group theoretic aspect of the result. The idea of developing generator–conjugations to pseudo–conjugations was suggested by John McCarthy.

Let $F_n$ be the free group on letters $x_1, \ldots, x_n$. This set of generators will be fixed throughout the paper. An endomorphism $\sigma$ of $F_n$ is a pseudo–conjugation if $\sigma$ maps generators to conjugates of generators, more precisely, for each $1 \leq i \leq n$

$$\sigma(x_i) = w_{\sigma,i} x_{\rho\sigma(i)} w_{\sigma,i}^{-1}$$

where $w_{\sigma,i}$ is a word in $F_n$ and $\rho\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ is a function. If $\rho\sigma$ is the identity, the endomorphism $\sigma$ is called a generator–conjugation. By the properties of free groups, it is easy to see that $\sigma$ determines $\rho\sigma$ uniquely. On the other hand, $w_{\sigma,i}$ is only determined by $\sigma$ up to a right multiplication by a power of $x_{\rho\sigma(i)}$. That is, we may replace $w_{\sigma,i}$, only by $w_{\sigma,i} x_{\rho\sigma(i)} m$ for some $m \in \mathbb{Z}$. $\sigma$ determines $w_{\sigma,i}$ uniquely if we assume that $w_{\sigma,i} x_{\rho\sigma(i)} w_{\sigma,i}^{-1}$ is freely reduced. From here on if $\sigma$ is a pseudo–conjugation then $w_{\sigma,i}$ will be defined by the free–reduction of $\sigma(x_i) = w_{\sigma,i} x_{\rho\sigma(i)} w_{\sigma,i}^{-1}$.

Let $\text{PC}(F_n)$ denote the monoid of all pseudo–conjugations of $F_n$. Let $T_n$ be the monoid of maps: $\{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. Then the map sending $\sigma$ in $\text{PC}(F_n)$ to $\rho\sigma$ in $T_n$ is a homomorphism.

Similarly let $\text{PCA}(F_n)$ denote the group of pseudo–conjugating automorphisms of $F_n$. Let $S_n$ be the group of permutations on $\{1, \ldots, n\}$.

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Passing to the abelianizations, one sees that \( p_\sigma \) in \( S_n \) is a homomorphism. Let \( \text{CA}(F_n) \) be the kernel of \( \pi \) and be called the group of generator-conjugating automorphisms.

For \( \rho \in S_n \), let \( \rho(p) \) defined to be an automorphism sending \( x_i \) to \( x_{p(i)} \) for \( 1 \leq i \leq n \). Then \( \rho \) is a monomorphism and

\[
1 \to \text{CA}(F_n) \to \text{PCA}(F_n) \to S_n \to 1
\]

becomes a split short exact sequence with splitting \( \rho \). Hence for a given \( \sigma \in \text{PCA}(F_n) \), there are a unique \( \rho \in S_n \) and a unique \( \alpha \in \text{CA}(F_n) \) such that \( \sigma = \rho \circ \alpha \). In fact, \( \alpha \) should be given to be \( \rho(\rho \sigma^{-1}) \circ \sigma \).

Let \( \text{AUF}(F_n) \) denote the group of automorphisms of \( F_n \). We recall that the \( n \)-th braid group \( B_n \) can be defined as the subgroup of \( \text{AUT}(F_n) \) that is generated by the automorphisms \( \beta_i \) for \( i = 1, \ldots, n-1 \) defined by \( \beta_i(x_i) = x_ix_{i+1}x_i^{-1} \), \( \beta_i(x_{i+1}) = x_i \) and \( \beta_i(x_j) = x_j \) for \( j \neq i, i+1 \). Over the fixed basis \( \{x_1, \ldots, x_n\} \), \( B_n \) is a subgroup of \( \text{PCA}(F_n) \).

In 1925, E. Artin characterized the element of \( B_n \) ([1], see also [2]). Let \( \beta \) be an endomorphism of \( F_n \). Then \( \beta \in B_n \) if and only if \( \beta \) satisfies the two conditions:

\[
\beta(x_i) = w_i x_{p(i)} w_i^{-1}, \quad 1 \leq i \leq n;
\]

\[
\beta(x_1 \cdots x_n) = x_1 \cdots x_n,
\]

where \( \rho \in S_n \) and \( w_i \) is a word in the letters \( x_1, \ldots, x_n \). This gives a simple condition which ensures that an endomorphism of \( F_n \) is an automorphism of \( F_n \). It is an interesting open problem to find a “simple” necessary and sufficient condition for this. We will present a simple necessary condition for a pseudo-conjugation to be an automorphism.

Now we discuss generators for \( \text{PCA}(F_n) \). \( \sigma \in \text{AUT}(F_n) \) is called a permutation if \( \sigma = \rho(p) \) for some \( \rho \in S_n \). \( \sigma \in \text{AUT}(F_n) \) is called a simple conjugation if there exists a pair of distinct integers \( i, j = 1, \ldots, n \) such that \( \sigma = \alpha_{i,j} \) where \( \alpha_{i,j} \) satisfies

\[
\alpha_{i,j}(x_i) = x_j x_i x_j^{-1};
\]

\[
\alpha_{i,j}(x_k) = x_k \quad \text{for} \quad k \neq i.
\]

**Theorem.** \( \text{CA}(F_n) \) is generated by simple conjugations.

In view of the above split short exact sequence, we have:

**Corollary.** \( \text{PCA}(F_n) \) is generated by permutations and simple conjugations.
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gations.

\( \sigma \in \text{AUT}(F_n) \) is a \textit{generalized simple braid} if there exists a pair of distinct integers \( i, j = 1, \ldots, n \) such that \( \sigma = \sigma_{i,j} \) where \( \sigma_{i,j} \) satisfies

\[
\begin{align*}
\sigma_{i,j}(x_i) &= x_i x_j x_i^{-1}; \\
\sigma_{i,j}(x_j) &= x_i; \\
\sigma_{i,j}(x_k) &= x_k, \quad k \neq i, j.
\end{align*}
\]

Artin's generators for \( B_n \) are just \( \sigma_{i,i+1} \) for \( 1 \leq i \leq n-1 \). Let \((i, j)\) be the transposition of \( i \) and \( j \) in \( S_n \). Then

\[
\alpha_{i,j} = (i, j) \circ \sigma_{i,j} \quad \text{for} \quad 1 \leq i \neq j \leq n
\]

Therefore we have:

**Corollary.** \( \text{PCA}(F_n) \) is generated by permutations and generalized simple braids.

Suppose \( \alpha \) is a generator-conjugation given by

\[
\alpha(x_i) = w_i x_i w_i^{-1} \quad \text{for} \quad i = 1, \ldots, n
\]

where \( w_i \) is a word and \( w_i x_i w_i^{-1} \) is freely reduced. It is convenient to write \( \alpha = (w_1, w_2, \ldots, w_n) \). This expression is unique. The simple conjugation \( \alpha_{i,j} \) is now written as \((e, \ldots, e, x_j, e, \ldots, e)\), and \( \alpha_{i,j}^{-1} \) is given by \((e, \ldots, e, x_j^{-1}, e, \ldots, e)\) where \( x_j^{\pm1} \) is in the \( i \)-th spot and \( e \) denotes the empty word.

**Lemma.** Let \( \alpha = (w_1, w_2, \ldots, w_n) \) be a generator-conjugation. If \( \alpha \) is an automorphism and is not the identity, then there is at least a pair \((j, k)\) of distinct integers among \( 1, 2, \ldots, n \) such that

\[
\omega_k = w_j x_j \varepsilon \overline{\omega}_k
\]

where \( \overline{\omega}_k \) is a reduced word which does not end with \( x_k \) or \( x_k^{-1} \) and \( \varepsilon = \pm 1 \).

Applying a permutation to this lemma, we have:

**Corollary.** Let \( \sigma \) be a pseudo-conjugation as given at the beginning of the paper. If \( \sigma \) is an automorphism and is not a permutation, then there is at least a pair \((j, k)\) of distinct integers among \( 1, 2, \ldots, n \) such that

\[
\omega_{\sigma,k} = \omega_{\sigma,j} x_{\rho_{\sigma}(j)} \varepsilon \overline{\omega}_k
\]

where \( \overline{\omega}_k \) is a reduced word which does not end with \( x_{\rho_{\sigma}(j)} \) or \( x_{\rho_{\sigma}(j)}^{-1} \)
and $\varepsilon = \pm 1$.

**Proof of Lemma.** Let $\alpha^{-1} = (u_1, u_2, ..., u_n)$. In fact, $u_i = \alpha^{-1}(w_i)$. Since $\alpha$ is not the identity, there is $1 \leq k \leq n$ such that $w_k$ is not the empty word and neither is $u_k$. If $u_k$ is given by the word $x_k^{e_1}x_k^{e_2}...x_k^{e_r}$, then

$$x_k = \alpha \circ \alpha^{-1}(x_k) = \left( \prod_{i=1}^{r} (w_k, x_k^{e_i}w_k^{-1}) \right) w_k x_k w_k^{-1} \left( \prod_{i=1}^{r} (w_k, x_k^{e_i}w_k^{-1}) \right)^{-1}.$$ 

Since $w_i x_i w_i^{-1}$ was reduced for each $i$, so is $w_k x_k^{e_i}w_k^{-1}$ for each $s$. Thus in the above product any cancelation must begin between one of pairs $(w_k^{-1}, w_k)$ or $(w_k^{-1}, w_{s+1})$ for $s=1, ..., r-1$. We now assume that Lemma is false. Then there is no way for any of the middle letters in the block, i.e., $x_k$ and $x_k^{e_1}, x_k^{e_2}, ..., x_k^{e_r}$ in the above product, to be canceled out. Since there is at least one letter among these middle letters which is different from $x_k$ or $x_k^{-1}$ (for example $x_k^{e_r}$), this is a contradiction.

When $\alpha = (w_1, w_2, ..., w_n)$ be a generator-conjugation, the length $l(\alpha)$ of $\alpha$ is defined as $l(\alpha) = l(w_1) + l(w_2) + ... + l(w_n)$ where $l(w_i)$ is the length of the reduced word $w_i$. It is clear that $l(\alpha) = 0$ if and only if $\alpha$ is the identity.

**Proof of Theorem.** Let $\alpha = (w_1, w_2, ..., w_n)$ be in $CA(F_n)$. Our proof is done by induction on $l(\alpha)$. Suppose that $\alpha$ is not the identity. Let $(j,k)$ be the pair given by Lemma. Since $w_k$ is reduced, $\bar{w}_k$ cannot start with $x_j^{-\varepsilon}$. Then $\alpha_\varepsilon \circ \alpha_{k,j^{-\varepsilon}} = (w_1, ..., w_{k-1}, w_j \bar{w}_k, w_{k+1}, ..., w_n)$. Thus $l(\alpha_\varepsilon \circ \alpha_{k,j^{-\varepsilon}})$ is strictly less than $l(\alpha)$. By induction, $\alpha$ can be made into the identity by composing $\alpha_{k,j^{-\varepsilon}}$'s on the right.

Lemma and the proof of Theorem give an effective method for determining whether a pseudo-conjugation is an automorphism.

Let $F_3$ be the free group on letters $x, y, z$. Define a pseudo-conjugation $\sigma : F_3 \rightarrow F_3$ by $\sigma(x) = yxy^{-1}y^{-1}$, $\sigma(y) = x$, $\sigma(z) = z$. After permuting $x$ and $y$, we have a generator-conjugation $(yx, e, e)$. By Lemma, this is not an automorphism.

For another example, let $\alpha = (z^{-1}y, x, xyz^{-1})$ be a generator-conjugation on $F_3$. Noticing $w_3 = w_2 y x^{-1}$, we have $\alpha \circ \alpha_{3,2}^{-1} = (z^{-1}y, x, e)$ Applying the same technique again, $\alpha \circ \alpha_{3,2}^{-1} \circ \alpha_{1,3} = (y, x, e)$. Now this
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conjugation does not satisfy the condition of Lemma and hence $\alpha$ was 
not an automorphism.

Theorem and its corollaries say that every automorphism in $\text{CA}(F_n)$ 
or in $\text{PCA}(F_n)$ can be written as a product of $\alpha_{i,j}$'s or a product of 
$\alpha_{i,j}$'s and a permutation. However the product is by no means unique. 
For example, $\alpha_{i,j}\circ \alpha_{k,j}=\alpha_{k,j}\circ \alpha_{i,j}$. The problem finding the complete 
set of relations among these generators of $\text{CA}(F_n)$ seems interesting. 
In [3], the quotient group of $\text{CA}(F_n)$ by inner automorphisms was 
useful and hence the same problem for this group needs some attention. 
These problems will be considered in sequent papers.

References

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Korea Institute of Technology
Taejon 302–338, Korea