Bull. Korean Math. Soc. 25(1988), No. 2, pp. 267-273

BEST APPROXIMATION BY COMPACT OPERATORS

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1. Introduction

Let J be a closed subspace of a Banach space X. An element x in X is said to have a best approximation in J if there is an element y in J such that $\|x-y\|=\inf\{\|x-j\| ; j\in J\}$. J is called a proximinal subspace of X if every x in X has a best approximation in J. It is known that a Banach space X is reflexive if and only if every closed subspace of X is proximinal [15]. If J is an M-ideal in X, then J is proximinal in X [8, 12, 19] and for each $x\in X\backslash J$ the set of best approximations in J of x algebraically spans J [10].

Many authors have studied the problem of determining those Banach spaces X and Y for which $K(X, Y)$, the space of compact linear operators from X to Y, is proximinal in $L(X, Y)$, the space of bounded linear operators from X to Y. If $X=Y$, we will write $L(X)$ (resp. $K(X)$) for $L(X, X)$ (resp. $K(X, X)$). Several sufficient conditions for the proximinality of $K(X, Y)$ in $L(X, Y)$ are known [2, 8, 11, 16, 19], and for certain spaces X and Y the questions whether $K(X, Y)$ are proximinal in $L(X, Y)$ are answered [2, 4-7, 11, 13, 16, 17, 19, 20]. For example, $K(c_0), K(l^1)$ and $K(l^p, l^q)$ for $1<p, q<\infty$ are proximinal in the corresponding spaces of operators [7, 11, 13] while if $X=l^\infty$ or $L^p(0,1)$ $(1\leq p\leq\infty, p\neq 2)$ $K(X)$ is not proximinal in $L(X)$ [6, 4].

Axler, Berg, Jewell and Shields [2] introduced a condition (called the Basic Inequality) on X and a condition on an operator $T\in L(X)$ which insure the existence of a best approximation in $K(X)$ of T. They [2] proved that if X is a Banach space which satisfies the Basic Inequality and which has a shrinking basis, then every T in $L(X)$ has a best approximation in $K(X)$. In particular, since l^p $(1<p<\infty)$ satisfies the Basic Inequality [2; Theorem 2], $K(l^p)$ is proximinal in

Received April 29, 1988.
Supported by the Ministry of Education, 1987.

$L(l^p)$.

In this paper we consider a pair (X, Y) of Banach spaces and use the method introduced in [2] to obtain best compact approximations of operators in $L(X, Y)$, In Corollary 6 we will prove that if X and Y are closed subspaces of l^p and l^q $(1<p\leq q<\infty)$, respectively, and if $T\in L(X, Y)$ is in the closure of $K(X, Y)$ in the strong operator topology (SOT in short), then T has a best approximation in $K(X, Y)$. Thus if either X or Y has the compact approximation property, then $K(X, Y)$ is proximinal in $L(X, Y)$.

2. The existence of the best approximation in $K(X, Y)$

Motivated by a definition in [2] we say that a pair (X, Y) of Banach spaces satisfies the *Basic Inequality* if for each T in $L(X, Y)$ and each sequence $\{A_n\}$ in $L(X, Y)$ such that $A_n\to 0$ (SOT) and $A_n^*\to 0$ (SOT) the following is true: for each $\varepsilon>0$, there exists N such that

$$\|T+A_N\|\leq\varepsilon+\max\{\|T\|, \|T\|_e+\|A_N\|\}$$

where $\|T\|_e=\inf\{\|T-K\|; K\in K(X, Y)\}$.

Then we have the following theorem which is really a restatement of Theorem 1 in [2] in an $L(X, Y)$ context.

THEOREM 1. *Suppose a pair (X, Y) of Banach spaces satisfies the Basic Inequality. Let $T\in L(X, Y)\backslash K(X, Y)$ and let $\{T_n\}$ be a sequence in $K(X, Y)$ such that $T_n\to T$ (SOT) and $T_n^*\to T^*$ (SOT). Then there exists a sequence $\{a_n\}$ of non-negative real numbers such that $\sum a_n=1$ and $\|T\|_e=\|T-K\|$, where $K=\sum a_nT_n$.*

Proof. The proof of Theorem 1 in [2] proves this by simply replacing operators in $L(X)$ by operators in $L(X, Y)$.

The proof of the next theorem is a minor modification of the proof of Theorem 2 in [2].

THEOREM 2. *If $1<p\leq q<\infty$ and X is a closed subspace of l^p, then (X, l^q) satisfies the Basic Inequaity.*

Proof. Let $T\in L(X,l^q)$ and let $\{A_n\}$ be a sequence in $L(X,l^q)$ such that $A_n\to 0$ (SOT) and $A_n^*\to 0$ (SOT). Let $\alpha_n=\max\{\|T\|, \|T\|_e+\|A_n\|\}$. If (X,l^q) did not satisfy the Basic Inequality, then there would exist $\varepsilon>0$ and a sequence $\{x_n\}$ of unit vectors in X such that

(1) $$\|(T+A_n)x_n\|>\varepsilon+\alpha_n \text{ for all } n.$$

Since B_X (the closed unit ball of X) is weakly compact, by passing to a subsequence if necessary we may assume that $x_n\to x\in B_X$ weakly. Set $x_n=x+y_n$ where $y_n\to 0$ weakly. Let P_n (resp. Q_n) be the projection of l^p (resp. l^q) onto the span of the first n vectors in the unit vector basis.

We claim that $\limsup_{n\to\infty}\{\|x\|^q+\|y_n\|^p\}\le 1$ and $\|T\|_e=\lim_{n\to\infty}\|(I-Q_n)T\|$.
Given $\eta>0$, choose m so that $\|(I-P_m)x\|<\eta$.
Since $y_n\to 0$ weakly, $\|P_my_n\|<\eta$ for all large n.

Thus
$$\begin{aligned}\|x\|^p+\|y_n\|^p&=\|P_mx\|^p+\|(I-P_m)y_n\|^p+\|(I-P_m)x\|^p+\|P_my_n\|^p\\&<\|P_mx+(I-P_m)y_n\|^p+2\eta^p\\&=\|x+y_n-(I-P_m)x-P_my_n\|^p+2\eta^p\\&\le(\|x_n\|+\|(I-P_m)x\|+\|P_my_n\|)^p+2\eta^p\\&<(1+2\eta)^p+2\eta^p \quad \text{for all sufficiently large } n.\end{aligned}$$
Since η is arbitrary, $\limsup_{n\to\infty}\{\|x\|^p+\|y_n\|^p\}\le 1$.

Since $\|T-K\|\ge\|(I-Q_n)(T-K)\|\ge\|(I-Q_n)T\|-\|(I-Q_n)K\|$ and $\|(I-Q_n)K\|\to 0$ for each $K\in K(X,l^q)$, we have that

$$\|T-K\|\ge\limsup_{n\to\infty}\|(I-Q_n)T\| \text{ and hence } \|T\|_e\ge\limsup_{n\to\infty}\|(I-Q_n)T\|.$$

Obviously $\liminf_{n\to\infty}\|T-Q_nT\|\ge\|T\|_e$. Hence $\lim_{n\to\infty}\|(I-Q_n)T\|=\|T\|_e$.

Now fix $0<\delta<1$ and choose M so that $\|(I-Q_M)x\|<\delta$ and $\|(I-Q_M)T\|<\|T\|_e+\delta$. Since $y_n\to 0$ weakly and Q_MT is compact, $\|Q_MTy_n\|\to 0$. Since Q_M^* is compact, $A_n^*\to 0$ (SOT) and $A_n\to 0$ (SOT), it follows that $\|Q_MA_n\|=\|A_n^*Q_M^*\|\to 0$ and $\|A_nx\|\to 0$.
Thus for all sufficiently large n.

(2) $$\begin{aligned}\|(T+A_n)x_n\|&\le\|Q_MTx+(I-Q_M)(Ty_n+A_ny_n)\|\\&\quad+\|(I-Q_M)Tx+Q_M(Ty_n+A_ny_n)+A_nx\|\\&<\|Q_MTx+(I-Q_M)(Ty_n+A_ny_n)\|+4\delta.\end{aligned}$$

Since Q_MTx and $(I-Q_M)(Ty_n+A_ny_n)$ have disjoint sppports, we have

$$(3)\quad \|Q_M Tx+(I-Q_M)(Ty_n+A_n y_n)\|^q$$
$$=\|Q_M Tx\|^q+\|(I-Q_M)(Ty_n+A_n y_n)\|^q$$
$$\leq \|T\|^q\|x\|^q+(\|(I-Q_M)T\|+\|A_n\|)^q\|y_n\|^q$$
$$\leq (\alpha_n+\delta)^q(\|y\|^q+\|y_n\|^q).$$

Since $\limsup_{n\to\infty}\ \{\|x\|^p+\|y_n\|^p\}\leq 1$, $\|y_n\|\leq 1$ and hence

$$\|x\|^q+\|y_n\|^q\leq\|x\|^p+\|y_n\|^p\leq 1+\delta \text{ for all sufficiently large } n.$$

Thus from (2) and (3) we get that

$$\|(T+A_n)x_n\|<(\alpha_n+\delta)(1+\delta)^{1/q}+4\delta \text{ for all sufficiently large } n.$$

Since $\delta>0$ is arbitrary, this contradicts to (1).

THEOREM 3 *Suppose X and Y are subspaces of l^p and l^q $(1<p\leq q<\infty)$, respectively. Then (X, Y) satisfies the Basic Inequality.*

Proof. First observe that if $T\in L(X, Y)$ and $J: Y\to l^q$ is the inclusion, then for each y^* in $(l^q)^*$, $(JT)^*y^*=T^*(y^*|_Y)$, where $y^*|_Y$ is the restriction of y^* to Y.

Suppose $\{A_n\}$ is a sequence in $L(X, Y)$ such that $A_n\to 0$ (SOT) and $A_n^*\to 0$ (SOT). Then $JA_n\to 0$ (SOT) and $(JA_n)^*\to 0$ (SOT). Since (X, l^q) satisfies the Basic Inequality, given T in $L(X, Y)$ and $\varepsilon>0$, there exists a positive integer N such that

$$\|JT+JA_N\|\ \leq\ \varepsilon+\max\{\|JT\|,\ \|JT\|_e+\|JA_N\|\}$$

where $\|JT\|_e\ =\ \inf\ \{\|JT-K\|\ ;\ K\in K(X, l^q)\}$
Since $\|JT\|_e\ \leq\ \|T\|_e$ and $\|JS\|=\|S\|$ for every S in $L(X, Y)$, we have

$$\|T+A_N\|\ \leq\ \varepsilon+\max\{\|T\|, \|T\|_e+\|A_N\|\}.$$

THEOREM 4. *Suppose X is a separable reflexive Banach space and Y is a Banach space with the separable dual space. If (X, Y) satisfies the Basic Inequality and $T\in L(X, Y)$ is in the closure of $K(X, Y)$ in the strong operator topology, then T has a best approximation in $K(X, Y)$.*

Proof. Suppose $T\in L(X, Y)$ is in the closure of $K(X, Y)$ in the strong operator topology and suppose $\{K_\alpha\}$ is a net in $K(X, Y)$ such that $K_\alpha\to T$ (SOT). Then $\{K_\alpha\}$ is uniformly bounded. We choose a countable dense subset $D=\{x_i\,;\, i=1, 2, 3, \cdots\}$ of X. For each n let $D_n=\{x_1, x_2, \cdots, x_n\}$ and choose T_n from $\{K_\alpha\}$ such that $\|T_n x-Tx\| < 1/n$ for all x in D_n. Then $\{T_n\}$ converges to T (SOT).

Since for each x in X and each y^* in Y^* $(T_n{}^*y^*)x=y^*(T_n x)$ converges to $y^*(Tx)=(T^*y^*)x$ and since X is reflexive, $T_n{}^*y^*$ converges to T^*y^* weakly for every y^* in Y^*.

Let $E=\{y_j{}^*\,;\, j=1, 2, 3, \cdots\}$ be a countable dense subset of Y^* and choose a sequence $\{S_{1n}\}_{n=1}^{\infty}$ so that each S_{1n} is a convex combination of $\{T_j\}_{j=n}^{\infty}$ and $S_{1n}{}^*y_1{}^* \to T^*y_1{}^*$ in norm. Next choose a sequence $\{S_{2n}\}_{n=1}^{\infty}$ so that S_{2n} is a convex combination of $\{S_{1j}\}_{j=n}^{\infty}$ and $S_{2n}{}^*y_2{}^*\to T^*y_2{}^*$ in norm. We continue the process in an obvious manner and let $S_n=S_{nn}$. Then $S_n{}^*y_j{}^*\to T^*y_j{}^*$ as $n\to\infty$ for each j and hence $S_n{}^*\to T^*$ (SOT). Obviously each S_n is in $K(X, Y)$ and $S_n\to T$ (SOT). By Theorem 1 and Theorem 3, T has a best approximation in $K(X, Y)$.

Recall that a Banach space X is said to have the compact approximation property if the identity operator on X is in the closure of $K(X)$ in the topology of uniform convergence on compact subsets of X.

COROLLARY 5. *Let X and Y be as in Theorem 4 and let (X, Y) satisfy the Basic Inequality. If either X or Y satisfies the the compact approximation property, then $K(X, Y)$ is proximinal in $L(X, Y)$.*

Proof. Suppose X has the compact approximation property. Let $0\neq T\in L(X, Y)$ and let K be a compact subset of X. Then for any $\varepsilon > 0$ there exists a compact operator T on X such that $\|T_1x-x\| < \varepsilon/\|T\|$ for all $x\in K$. Now $TT_1\in K(X, Y)$ and $\|TT_1x-Tx\| < \varepsilon$ for all $x\in K$. This shows that $K(X, Y)$ is dense in $L(X, Y)$ in the topology of uniform convergence on compact subsets in X and hence in the strong operator topology. By Theorem 4, $K(X, Y)$ is proximinal in $L(X, Y)$. The proof of the other case is similar.

From Theorem 3 and Corollary 5, we have the following corollary.

COROLLARY 6. *Let X and Y be closed subspaces of l^p and l^q $(1<p\leq q<\infty)$, respectively. If either X or Y satisfies the compact approximation property, then $K(X, Y)$ is proximinal in $L(X, Y)$.*

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