A STABILITY IN TOPOLOGICAL DYNAMICS

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1. Introduction

Theorem. Let \((X, \phi)\) be a flow whose phase space \(X\) is a locally compact metric space. Then a compact invariant subset \(M\) of \(X\) is asymptotically stable if and only if there exists a continuous nonnegative real valued function \(f\) defined on an invariant neighborhood \(U\) of \(M\) such that \(f\) vanishes exactly on \(M\), and that \(f(\phi t x) = e^{-t} f(x)\) for all points \(x\) of \(U\) and real numbers \(t\) [1].

In this paper we introduce the concept of a \(c\)-first countable space which is a more general concept than that of a metric space, and extend the above theorem to the case that the phase space \(X\) is \(c\)-first countable and locally compact. All spaces are assumed to be Hausdorff.

2. \(C\)-first countable spaces.

Definition. A space \(X\) is said to be \(c\)-first countable if for each compact subset \(K\) of \(X\) the quotient space \(X/K\) is first countable.

Let \(X\) be a \(c\)-first countable space. Given any compact subset \(K\) of \(X\), there exists a family \(\mathcal{U}\) consisting of countably many neighborhoods of \(K\) such that every neighborhood of \(K\) contains some member of \(\mathcal{U}\). Such a family \(\mathcal{U}\) will be called a countable neighborhood base of \(K\).

Theorem 2.1 Every second countable space is \(c\)-first countable.

Proof. Let \(X\) be a second countable space. There exists a countable basis \(\mathcal{B}\) for \(X\). Given any compact subset \(K\) of \(X\), let \(\mathcal{U}\) be the family of all neighborhoods of \(K\) which are finite unions of members of \(\mathcal{B}\). Then \(\mathcal{U}\) is a countable neighborhood base of \(K\). Thus \(X\) is \(c\)-first countable.
The converse of the above theorem is not true as shown by uncountable discrete spaces. Clearly, every $c$–first countable space is first countable but its converse does not hold.

**Example 2.1.** Let $X_0 = \{(x, 0) : x \in \mathbb{R}\}$ and $X_1 = \{(x, 1) : x \in \mathbb{R}\}$ be two subsets of the plane $\mathbb{R}^2$. We take a basis $\mathcal{B}$ for the topology on the set $X = X_0 \cup X_1$ as follow:

$$\mathcal{B} = \{(x, 1) : x \in \mathbb{R}\} \cup \{B(x, r) : x \in \mathbb{R}, r > 0\}$$

where $B(x, r) = \{(y, 0) : |x-y| < r\} \cup \{(y, 1) : 0 < |x-y| < r\}$. It is clear that $X$ is first countable. We claim that $X$ is not $c$–first countable. Let us choose a compact subset $K = \{(x, 0) : x \in I\}$ of $X$ where $I$ is the unit interval. For each neighborhood $U$ of $K$, let $I(U) = \{x \in I : (x, 1) \notin U\}$. Suppose that $I(U)$ is infinite for some neighborhood $U$ of $K$. $I(U)$ has a cluster point, say $y$, in $I$. Since $(y, 0) \in K \subset U$, there exists a number $r > 0$ such that $B(y, r) \subset U$. Since $y$ is a cluster point of $I(U)$, there is a number $z \in I(U)$ such that $0 < |y-z| < r$. Since $(z, 1) \in B(y, r) \subset U$, we have a contradiction. Thus $I(U)$ is finite for all neighborhoods $U$ of $K$. Let $U_1, U_2, U_3, \ldots$ be neighborhoods of $K$. Since $I(U_n)$ is finite for all $n$, $A = \bigcup_{n=1}^{\infty} I(U_n)$ is countable. Thus there is a number $w \in I - A$. Let $V = X_0 \cup \{(x, 1) : x \neq w\}$. Then $V$ is a neighborhood of $K$ and $U_n \subset V$ for all $n$. Thus there is no countable neighborhood base of $K$. Hence $X$ is not $c$–first countable.

**Theorem 2.2.** Every metric space is $c$–first countable.

**Proof.** Let $(X, d)$ be a metric space. Given any compact subset $K$ of $X$, it is easy to show that the family $\{B(K, \frac{1}{n}) : n = 1, 2, 3, \ldots\}$ is a countable neighborhood base of $K$, where $B(K, \frac{1}{n}) = \{x \in X : d(K, x) < \frac{1}{n}\}$. Thus $X$ is $c$–first countable.

The converse of the above theorem is not true. The following example shows that there exists a $c$–first countable and locally compact space which is not a metric space.

**Example 2.2.** For each irrational $x$, we choose a sequence $(x_n)$ of
rational sequence topology \( \overline{\sigma} \) on \( \mathbb{R} \) is then defined by declaring each rational open, and selecting the sets \( U_n(x) = \{x_i : i=n, n+1, n+2, \ldots \} \cup \{x\} \) as a basis for the irrational point \( x \). The space \( (\mathbb{R}, \overline{\sigma}) \) is Hausdorff, locally compact and not metrizable [2]. We will show that \( (\mathbb{R}, \overline{\sigma}) \) is \( c \)-first countable. Let \( K \) be a compact subset of \( \mathbb{R} \). If \( K-Q \) is infinite, where \( Q \) is the set of rationals, then the open cover \( \{U_i(x) : x \in K-Q \} \cup \{Q\} \) of \( K \) has no finite subcover, this is a contradiction. Thus \( K-Q \) is finite, say \( K-Q = \{x^1, x^2, \ldots, x^m\} \). Let \( U \) be a neighborhood of \( K \). For each \( i=1, 2, \ldots, m \), since \( x^i \in K-Q \subset U-Q \), there is an \( n_i \) such that \( U_{n_i}(x^i) \subset U \).

Let \( N=\max n_i \). Then \( \bigcup_{i=1}^{n} U_{n_i}(x^i) \cup (K \cap Q) \subset U \). Thus \( \left\{ \bigcup_{i=1}^{n} U_{n_i}(x^i) \cup (K \cap Q) : n=1, 2, \ldots \right\} \) is a countable neighborhood base of \( K \). Hence \( (\mathbb{R}, \overline{\sigma}) \) is \( c \)-first countable.

**Lemma 2.1.** Let \( X \) be a \( c \)-first countable and locally compact space, and let \( K \) be a compact subset of \( X \). For each neighborhood \( U \) of \( K \), there exists a countable neighborhood base \( \{U(r) : r \in D\} \) of \( K \) such that

1. \( U(1) = U \), and that
2. if \( r_1 < r_2 \), then \( U(r_1) \subset U(r_2) \)

where \( D \) is the set of all rationals of form \( \frac{k}{2^n}, 0 < \frac{k}{2^n} \leq 1 \).

**Proof.** Let us show that for each \( r \in D \) we can associate a neighborhood \( U(r) \) of \( K \) satisfying the above conditions (1) and (2). We proceed by induction on exponent of dyadic fractions, letting \( \mathcal{U}_n = \left\{ U\left(\frac{k}{2^n}\right) : k=1, 2, \ldots, 2^n \right\} \). There exists a countable neighborhood base \( \{V_m : m=1, 2, \ldots\} \) of \( K \). We may assume that \( V_1 \supset V_2 \supset \ldots \) and \( V_1 \) compact. There is an \( m_1 \) such that \( V_{m_1} \subset K \). \( \mathcal{U}_1 \) consists of \( U\left(\frac{1}{2}\right) = V_{m_1} \) and \( U(1) = U \). Assume \( \mathcal{U}_{n-1} \) constructed. Note that only \( U\left(\frac{k}{2^n}\right) \) for odd \( k \) requires definition. There is an \( m_n > m_{n-1} \) such that \( \overline{V}_{m_n} \subset U\left(\frac{1}{2^n}\right) \). We define \( U\left(\frac{1}{2^n}\right) = V_{m_n} \). For each odd \( k \neq 1 \), we have from \( \mathcal{U}_{n-1} \) that \( U\left(\frac{k-1}{2^n}\right) \subset U\left(\frac{k+1}{2^n}\right) \), so we define \( U\left(\frac{k}{2^n}\right) \) to be an open set \( V \) satisfying
\begin{equation*}
U\left(\frac{k-1}{2^n}\right) \subset V \subset \bar{V} \subset U\left(\frac{k+1}{2^n}\right)
\end{equation*}

and \(\bar{V}\) compact. This completes inductive step. Given any neighborhood \(W\) of \(K\), there is an \(n\) such that \(V_{m_n} = U\left(\frac{1}{2^n}\right) \subset W\). Thus the family \(\{U(r) : r \in D\}\) is a countable neighborhood base of \(K\).

**Theorem 2.3.** Let \(X\) be a locally compact space. Then \(X\) is \(c\)-first countable if and only if for each compact subset \(K\) of \(X\) there exists a continuous nonnegative real valued function \(f\) defined on \(X\) such that \(f\) vanishes exactly on \(K\).

**Proof.** \((\Rightarrow)\) By Lemma 2.1, there exists a countable neighborhood base \(\{U(r) : r \in D\}\) such that \(U(1) = X\), and that if \(r_1 < r_2\) then \(\overline{U(r_1)} \subset U(r_2)\). Define a function \(f : X \to \mathbb{R}^+\) by \(f(x) = \inf\{r \in D : x \in U(r)\}\). Clearly, \(0 \leq f \leq 1\). It is easy to show that \(f\) vanishes exactly on \(K\). Given any \(\varepsilon > 0\), we can choose an \(r \in D\) such that \(r < \varepsilon\). Since \(f(U(r)) \subset (-\varepsilon, \varepsilon)\), \(f\) is continuous on \(K\). We will show that \(f\) is continuous at \(x \in X - K\). There are two possibilities:

Case 1. \(f(x) < 1\); Given any \(\varepsilon > 0\), we can choose \(r_1\) and \(r_2\) in \(D\) such that \(f(x) - \varepsilon < r_1 < f(x) < r_2 < f(x) + \varepsilon\). Then \(U(r_2) - \overline{U(r_1)}\) is a neighborhood of \(x\) and \(f(U(r_2) - \overline{U(r_1)}) \subset (f(x) - \varepsilon, f(x) + \varepsilon)\).

Case 2. \(f(x) = 1\); Given any \(\varepsilon > 0\), there exists a number \(r \in D\) such that \(1 - \varepsilon < r < 1\). Then \(X - \overline{U(r)}\) is a neighborhood of \(x\) and \(f(X - \overline{U(x)}) \subset (1 - \varepsilon, 1 + \varepsilon)\). Thus \(f\) is continuous.

\((\Leftarrow)\) There exists a neighborhood \(U\) of \(K\) such that \(\overline{U}\) is compact. For each positive integer \(n\), the set \(U_n = f^{-1}\left[0, \frac{1}{n}\right] \cap U\) is a neighborhood of \(K\). Given any neighborhood \(V\) of \(K\), suppose that \(U_n \not\subset V\) for all \(n\). For each \(n\), we can choose an \(x_n \in U_n - V\). Since \(\overline{U}\) is compact, the sequence \((x_n)\) in \(\overline{U}\) has a convergent subsequence. Let \(x_n \to x\). It is clear that \(x \in X - V\) and \(f(x_n) \to f(x)\). Since \(f(x_n) < \frac{1}{n}\) for all \(n\), \(f(x_n) \to 0\). Thus \(f(x) = 0\) and so \(x \in K\). This is a contradiction. So \(U_n \subset V\) for some \(n\). Hence the family \(\{U_n : n = 1, 2, \ldots\}\) is a countable neighborhood base of \(K\).
3. Asymptotic stability

Throughout this section $(X, \varphi)$ is a flow whose phase space $X$ is $\sigma$-first countable and locally compact.

For a point $x$ of $X$, the positive (negative) limit set $L^+(x)$ ($L^-(x)$) of $x$ defined by

$$L^+(x) = \bigcap_{t \in \mathbb{R}^+} x[t, \infty) \quad (L^-(x) = \bigcap_{t \in \mathbb{R}^-} x(-\infty, t])$$

where $\mathbb{R}^+$ ($\mathbb{R}^-$) denotes the set of nonnegative (nonpositive) real numbers. It is easy to show that $y \in L^+(x)$ ($L^-(x)$) if and only if there is a sequence $(t_n)$ in $\mathbb{R}^+$ ($\mathbb{R}^-$) such that $t_n \to \infty$ ($-\infty$) and $x_{t_n} \to y$. Obviously, the set $L^+(x)$ ($L^-(x)$) is invariant. Furthermore, the set $L^+(x)$ ($L^-(x)$) is nonempty whenever $\overline{x \mathbb{R}^+}$ ($\overline{x \mathbb{R}^-}$) is compact. A subset $M$ of $X$ is said to be stable if for each neighborhood $U$ of $M$, there exists a neighborhood $V$ of $M$ such that $V \mathbb{R}^+ \subset U$. It is clear that a stable set is positively invariant. For a subset $M$ of $X$, the region of attraction $A(M)$ is defined by $A(M) = \{x \in X : L^+(x) \neq \emptyset \subset M\}$. Note that $A(M)$ is invariant. A subset $M$ of $X$ is called an attractor if the set $A(M)$ is a neighborhood of $M$. When a subset $M$ of $X$ is stable and an attractor, the set $M$ is said to be asymptotically stable.

**Lemma 3.1** Let $M$ be a compact subset of $X$. Then $x \in A(M)$ if and only if for each neighborhood $U$ of $M$ there exists a $t \in \mathbb{R}^+$ such that $x[t, \infty) \subset U$.

**Proof.** ($\Rightarrow$) Let $x \in A(M)$ and $U$ a neighborhood of $M$. We can choose a neighborhood $V$ of $M$ such that $\overline{V} \subset U$ and $\overline{V}$ compact. Suppose that for each $t \in \mathbb{R}^+$ there is an $s \geq t$ such that $x_s \notin U$. Then there is an $r_1 \geq 1$ such that $x_{r_1} \in X - U \subset X - \overline{V}$. Since $x \in A(M)$, there exists a $t_1 > r_1$ such that $x_{t_1} \in V$. We can choose an $s_1$ such that $r_1 < s_1 < t_1$ and $x_{s_1} \in \partial V$ where $\partial V$ is the boundary of $V$. By the same way we can choose $r_2, t_2$ and $s_2$ such that

$$r_2 \geq \max(2, t_1), \quad x_{r_2} \in X - \overline{V}, \quad x_{t_2} \in V, \quad r_2 < s_2 < t_2 \quad \text{and} \quad x_{s_2} \in \partial V,$$

and so on. Thus we obtain a sequence $(s_n)$ in $\mathbb{R}^+$ such that $s_n \to \infty$ and $x_{s_n} \in \partial V$ for all $n$. Since $\partial V$ is compact, the sequence $(x_{s_n})$ has a convergent subsequence. Let $x_{s_n} \to z \in \partial V$. Since $z \in L^+(x) \subset M \subset V$, we have a contradiction. Thus there is a $t \in \mathbb{R}^+$ such that $x[t, \infty) \subset U$. 


There exists a neighborhood $U$ of $M$ such that $\overline{U}$ is compact. We can choose a $t \in \mathbb{R}^+$ such that $x[t, \infty) \subset U$. Since
$$\overline{xR^+} = x[0, t] \cup \overline{x[t, \infty)} \subset x[0, t] \cup \overline{U},$$
$xR^+$ is compact. Thus $L^+(x) \neq \phi$. To show $L^+(x) \subset M$, suppose that there exists an $y \in L^+(x) - M$. There are neighborhoods $V$ of $M$ and $W$ of $y$ such $V \cap W = \phi$. We can choose a $t \in \mathbb{R}^+$ such that $x[t, \infty) \subset V$. Since $W \cap x[t, \infty) = \phi$, $y \notin \overline{x[t, \infty)}$ and so $y \notin L^+(x)$. This is a contradiction. Thus $L^+(x) \subset M$. Hence $x \in A(M)$.

**Lemma 3.2** Let a compact subset $M$ of $X$ be asymptotically stable and $U$ a neighborhood of $M$. For any point $x$ of $A(M)$, if $xR^+ \subset U$, then there exists a neighborhood $V$ of $x$ such that $VR^+ \subset U$.

**Proof.** Since $M$ is stable, there is a neighborhood $U_1$ of $M$ such that $U_1 \cap U^+ \subset U$. By Lemma 3.1, there is an $s \in \mathbb{R}^+$ such that $x[s, \infty) \subset U_1$. We can choose a neighborhood $W_1$ of $x$ such that $W_1 \subset U_1$. For each $t \in [0, s]$, since $xt \subset U$, there exist neighborhoods $V_t$ of $x$ and $I_t$ of $t$ such that $V_t \cap I_t \subset U$. There are finitely many $0 \leq t_1, t_2, \ldots, t_n \leq s$ such that $[0, s] \subset \bigcup_{i=1}^{n} I_t$. Let $W_2 = \bigcap_{i=1}^{n} V_t$. Then $W_2$ is a neighborhood of $x$. Given any $y \in W_2$ and $t \in [0, s]$, since $t \in I_t$ for some $i$, $yt \in V_t \cap I_t \subset U$. Thus $W_2 \subset [0, s] \subset U$. Let $V = W_1 \cap W_2$. Then $V$ is a neighborhood of $x$. From the fact that

$$V[0, s] \subset W_2[0, s] \subset U \quad \text{and} \quad V[s, \infty) \subset W_1[s, \infty) = (W_1s) \cap U \subset U,$$
we have $VR^+ = V[0, s] \cup V[s, \infty) \subset U$.

**Lemma 3.3** Let $U$ be a neighborhood of a point $x$ of $X$. If $y$ is a point of $X$ and $yR^+ \notin U$, then there is a neighborhood $V$ of $y$ such that $zR^+ \notin U$ for all points $z$ of $V$.

**Proof.** There is a $t \in \mathbb{R}^+$ such that $yt \notin U$. Since $X - U$ is a neighborhood of $yt$, there exists a neighborhood $V$ of $y$ such that $Vt \subset X - U$. Then $V$ is a desired neighborhood.

**Theorem 3.1** Let $M$ be an asymptotically stable compact subset of $X$. Then there exists a continuous nonnegative real valued function $f$ defined on $A(M)$ such that $f$ vanishes exactly on $M$, and that $f(xt) < f(x)$ for all points $x$ of $A(M) - M$ and all positive real numbers $t$.

**Proof.** Let $D$ be the set of all rationals $s$ of form $\frac{k}{2^n}$, $0 < \frac{k}{2^n} \leq 1$. 
By Lemma 2.1, there exists a countable neighborhood base \( \{U(r) : r \in D\} \) of \( M \) satisfying

1. \( U(1) = A(M) \) and
2. if \( r_1 < r_2 \) then \( U(r_1) \subset U(r_2) \).

Define a function \( g : A(M) \to \mathbb{R}^+ \) by \( g(x) = \inf \{ r \in D : x^+ \subset U(r) \} \).

Clearly, \( 0 \leq g \leq 1 \). Let \( x \in M \). For any \( r \in D \), since \( x^+ \subset M \subset U(r) \), \( g(x) \leq r \). Thus \( g(x) = 0 \). Let \( x \in A(M) - M \). We can choose an \( r \in D \) such that \( x \not\in U(r) \). Then \( g(x) \geq r > 0 \). Thus \( g \) vanishes exactly on \( M \). Let us show that \( g \) is continuous on \( M \). Given any \( \varepsilon > 0 \), there exists a number \( r \in D \) such that \( r < \varepsilon \). Since \( M \) is stable, there exists a neighborhood \( V \) of \( M \) such that \( V^+ \subset U(r) \). Since \( g(V) \subset (-\varepsilon, \varepsilon) \), \( g \) is continuous on \( M \). We further show that \( g \) is continuous at each point \( x \) in \( A(M) - M \). There are two possibilities:

1. In case \( g(x) = 1 \), given any \( \varepsilon > 0 \), we can choose an \( r \in D \) such that \( 1 - \varepsilon < r < 1 \). Since \( x^+ \nsubseteq U(r) \), by Lemma 3.3, there is a neighborhood \( V \) of \( x \) such that \( y^+ \nsubseteq U(r) \) for all \( y \in V \). Then \( g(V) \subset (1-\varepsilon, 1+\varepsilon) \).

2. In case \( g(x) < 1 \), given any \( \varepsilon > 0 \), we choose \( r_1, r_2 \in D \) such that \( g(x) - \varepsilon < r_1 < g(x) < r_2 < g(x) + \varepsilon \). Since \( x^+ \subset U(r_2) \), there is a neighborhood \( V_1 \) of \( x \) such that \( V_1^+ \subset U(r_2) \) by Lemma 3.2. By Lemma 3.3, there exists a neighborhood \( V_2 \) of \( x \) such that \( y^+ \subset U(r_1) \) for all \( y \in V_2 \). Let \( V = V_1 \cap V_2 \). Then \( V \) is a neighborhood of \( x \) and \( g(V) \subset (g(x) - \varepsilon, g(x) + \varepsilon) \). Thus \( g \) is continuous. We claim that \( g(xt) \leq g(x) \) for all \( x \in A(M) \) and \( t \in \mathbb{R}^+ \). Suppose that \( g(xt) > g(x) \) for some \( x \in A(M) \) and \( t \in \mathbb{R}^+ \). We can choose an \( r \in D \) such that \( g(x) < r < g(xt) \). Since \( (xt)^+ = x[t, \infty) \subset x^+ \subset U(r) \), \( g(xt) \leq r \). This is a contradiction. Thus \( g(xt) \leq g(x) \) for all \( x \in A(M) \) and \( t \in \mathbb{R}^+ \).

Define a function \( f : A(M) \to \mathbb{R}^+ \) by

\[
 f(x) = \int_0^\infty e^{-s} g(xs) \, ds.
\]

Clearly, \( f \) is continuous and vanishes exactly on \( M \). It remains to prove that \( f(xt) < f(x) \) for all \( x \in A(M) - M \) and \( t > 0 \). Since \( g((xt)s) = g((xs)t) \leq g(xs) \) for all \( s \in \mathbb{R}^+ \), \( f(xt) \leq f(x) \). To rule out \( f(xt) = f(x) \), observe that in this case we must \( g(x(t+s)) = g((xs)t) = g(xs) \) for all \( s \in \mathbb{R}^+ \). In particular, letting \( s = 0, t, 2t, ... \), we get \( g(x(n)) = g(x) \), \( n = 1, 2, ... \). Given any \( r \in D \), since \( x \in A(M) \), by Lemma 3.1, there exists an \( s \in \mathbb{R}^+ \) such that \( x[s, \infty) \subset U(r) \). Since \( nt \to \infty \) as \( n \to \infty \), \( mt \to \infty \) as \( m \to \infty \).
\[ (x(\omega t)) R^+ = x[\omega t, \infty) \subset x[s, \infty) \subset U(r), \]

\[ g(x) = g(x(\omega t)) \leq r. \] Thus \( g(x) = 0. \) But as \( x \in A(M) - M, \) we must \( g(x) > 0, \) a contradiction. We have thus proved that \( f(\omega t) < f(x). \) The theorem is proved.

**Theorem 3.2** Let \( M \) be an asymptotically stable compact invariant subset of \( X. \) Then there exists a continuous function \( f : A(M) \to R^+ \) such that \( f \) vanishes exactly on \( M, \) and that \( f(\omega t) = e^{-tf(x)} \) for all \( x \in A(M) \) and all \( t \in R. \)

**Proof.** By Theorem 3.1, there exists a continuous function \( g : A(M) \to R^+ \) such that \( g \) vanishes exactly on \( M, \) and that \( g(\omega t) < g(x) \) for all \( x \in A(M) - M \) and all \( t > 0. \) Since \( A(M) \) is a neighborhood of \( M, \) we can choose a neighborhood \( U \) of \( M \) such that \( U \subset A(M) \) and \( U \) is compact. Set \( a = \text{min } g(\partial U). \) Clearly, \( a > 0. \) Let \( V = g^{-1}(0, a). \) Then \( V \) is a neighborhood of \( M. \) Suppose that \( \overline{V} \cap U \) and choose a point \( x \in \overline{V} - U. \) Since \( x \in V \subset g^{-1}(0, a) \subset A(M), \) there exists a number \( s > 0 \) such that \( x[s, \infty) \subset U \) by Lemma 3.1. Thus we can choose a \( t > 0 \) such that \( x \in \partial U. \) Since \( a \leq g(\omega t) < g(x) \leq a, \) we have a contradiction. This shows that \( \overline{V} \subset \partial U. \) We claim that \( \partial V \cap (\partial V) t = \emptyset \) for all \( t > 0. \) Suppose that \( \partial V \cap (\partial V) t \neq \emptyset \) for some \( t > 0. \) Then there exists an \( x \in \partial V \) such that \( x \in \partial V. \) Since \( \partial V \subset g^{-1}(a), \) \( a = g(\omega t) < g(x) = a. \) This is a contradiction. Thus \( \partial V \cap (\partial V) t = \emptyset \) for all \( t > 0. \) We will show that for every \( x \in A(M) - M, \) there is unique \( t \in R \) such that \( x \in \partial V. \) There are three possibilities:

1. In case \( x \notin \overline{V}, \) by Lemma 3.1, there is an \( s > 0 \) such that \( x[s, \infty) \subset V. \) Thus we can choose a \( t > 0 \) such that \( x \in \partial V. \)

2. In case \( x \in \partial V, \) \( x = x \in \partial V. \)

3. In case \( x \in V, \) assume that \( x \in V. \) Since \( \overline{x R} \subset \overline{V} \) is compact, \( L^-(x) \neq \emptyset. \) If \( L^-(x) \cap M = \emptyset, \) then we can choose an \( y \in L^-(x) \cap M. \) There exists a sequence \( (t_n) \in R^+ \) such that \( t_n \to 0 \) and \( x \in V. \) Since \( g(x) \leq g(x(t_n)) \) for all \( n, \) \( g(x) \leq g(y) = 0, \) this is a contradiction. Thus \( L^-(x) \cap M = \emptyset. \) Choose a point \( x \in L^-(x). \) Since \( L^+(x) \subset \overline{x R} \subset L^-(x), \) \( L^-(x) \cap M = \emptyset. \) But \( L^+(x) \) is nonempty and contained in \( M \) because of \( x \in L^-(x) \subset \overline{x R} \subset \overline{V} \subset A(M). \) This is a contradiction. Thus \( x R \not\subset V. \) Hence we can choose a \( t \in R \) such that \( x \in \partial V. \) The uniqueness of such \( t \) can be obtained from the fact that \( \partial V \cap (\partial V) t = \emptyset \) for all \( t > 0. \)
Define a function \( m : A(M) - M \rightarrow \mathbb{R} \) by \( x m (x) \in \partial V \). Let \( x \in A(M) - M \). Given any \( t \in \mathbb{R} \), since \( (x t) (m(x) - t) = x m (x) \in \partial V \), \( m(x t) = m(x) - t \). Thus \( m(x t) \rightarrow \pm \infty \) as \( t \rightarrow \mp \infty \). We will show that \( m \) is continuous. Given any \( x \in A(M) - M \) and \( \varepsilon > 0 \), since \( x (m(x) + \varepsilon) \in V \), \( W_1 = V (-m(x) - \varepsilon) \) is a neighborhood of \( x \). For all \( y \in W_1 \), \( y (m(x) + \varepsilon) \in V \) implies \( m(y) < m(x) + \varepsilon \). Since \( x (m(x) - \varepsilon) \in X - \bar{V} \), \( W_2 = (X - \bar{V}) (-m(x) + \varepsilon) \) is a neighborhood of \( x \). For all \( y \in W_2 \), \( y (m(x) - \varepsilon) \in X - \bar{V} \) implies \( m(x) - \varepsilon < m(y) \). Let \( W = W_1 \cap W_2 \). Then \( W \) is a neighborhood of \( x \) and \( m(x) - \varepsilon < m(y) < m(x) + \varepsilon \) for all \( y \in W \). Thus \( m \) is continuous.

Define a function \( f : A(M) \rightarrow \mathbb{R}^+ \) by
\[
  f(x) = \begin{cases} 
    e^{m(x)} & \text{if } x \in A(M) - M \\
    0 & \text{if } x \in M. 
  \end{cases}
\]

We will show that \( f \) is continuous. It is sufficient to show that \( f \) is continuous on \( M \). Suppose that there exists an \( \varepsilon > 0 \) such that \( f(U) \subset [0, \varepsilon) \) for all neighborhoods \( U \) of \( M \). There is a \( T \in \mathbb{R}^- \) such that \( e^T \leq \varepsilon \). For each neighborhood \( U \) of \( M \), \( f(U) \subset [0, e^T) \) and so \( m(U - M) \subset (-\infty, T) \). Since \( X \) is \( c \)–first countable, there is a countable neighborhood base \( \{ V_n : n = 1, 2, \ldots \} \) of \( M \). We may assume that \( V \supset V_1 \supset V_2 \supset \ldots \). For each \( n \), since \( m(V_n - M) \subset (-\infty, T) \), there is an \( x_n \in V_n - M \) such that \( T \leq m(x_n) \leq 0 \). There is a \( y \in M \) such that \( x_n \rightarrow y \). \( (m(x_n)) \) is a sequence in \( [T, 0] \). Since \( [T, 0] \) is compact, \( (m(x_n)) \) has a convergent subsequence. Let \( m(x_n) \rightarrow t \in [T, 0] \). Then \( x_n m(x_n) \rightarrow yt \in M \) and \( yt \in \partial V \). This is a contradiction. Thus for each \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( M \) such that \( f(U) \subset [0, \varepsilon) \). Hence \( f \) is continuous on \( M \). Clearly, \( f \) vanishes exactly on \( M \). For any \( x \in A(M) \) and \( t \in \mathbb{R} \),
\[
  f(x t) = e^{m(x t)} = e^{m(x) - t} = e^{-t} e^{m(x)} = e^{-t} f(x).
\]

Thus the theorem is proved.

**Lemma 3.4** Let \( M \) be a compact subset of \( X \), \( U \) an invariant neighborhood of \( M \) and \( f : U \rightarrow \mathbb{R}^+ \) a continuous function such that \( f \) vanishes exactly on \( M \) and \( f(x t) = e^{-t} f(x) \) for all \( x \in U \) and \( t \in \mathbb{R} \). If \( K \) is a compact positively invariant subset of \( U \) then \( K \) is contained in \( A(M) \).

**Proof.** Let \( x \in K \). Since \( x \mathbb{R}^+ \subset K \) is compact, \( L^+ (x) \neq \emptyset \). Let \( y \in L^+ (x) \). Take a \( t > 0 \). Since \( yt \in L^+ (x) \), there are sequence \( (t_n) \), \( (s_n) \) in \( \mathbb{R}^+ \) such that \( t_n \rightarrow \infty \), \( s_n \rightarrow \infty \), \( xt_n \rightarrow y \) and \( xs_n \rightarrow yt \). We may assume that \( t_n \geq s_n \) for all \( n \). Since \( f(x t_n) \leq f(xs_n) \), \( f(y) \leq f(yt) \). Since \( f(yt) \leq f(y) \),
f(y) = f(yt) = e^{-tf(y)}. Thus f(y) = 0 and so y \in M. Hence L^+(x) \subseteq M. Therefore x \in A(M).

**Theorem 3.3** Let M be a compact invariant subset of X. If there exists a continuous nonnegative real valued function f defined on an invariant neighborhood U of M such that f vanishes exactly on M, and that f(xt) = e^{-tf(x)} for all points x of U and all real numbers t, then M is asymptotically stable and U = A(M).

**Proof.** Given any neighborhood V of M, we can choose a neighborhood W_1 of M such that \overline{W_1} \subseteq U \cap V and \overline{W_1} is compact. Let a = \min f(\partial W_1). Then a > 0. Let W = f^{-1}[0, a). Then W \subseteq W_1 and W is a positively invariant neighborhood of M. Thus M is stable. We can choose a neighborhood V of M such that \overline{V} \subseteq U and \overline{V} is compact. Let a = \min f(\partial V). Then a > 0. Take a number r such that 0 < r < a, and let W = f^{-1}[0, r]. Then W \subseteq V and W is compact positively invariant. By Lemma 3.4, W \subseteq A(M). Given any x \in U, if x \in W, then x \in A(M), and if x \in W, then f(x) > r. There is a t > 0 such that f(xt) = e^{-tf(x)} = r. Since K = xR^+ \cup W = x[0, t] \cup W is a compact positively invariant subset of U, by Lemma 3.4, K \subseteq A(M) and so x \in A(M). Thus U \subseteq A(M). Given any x \in A(M), since U is a neighborhood of M, by Lemma 3.1, there is a t \in R^+ such that xt \in U. Since U is invariant, x = (xt)(-t) \in U. Hence A(M) = U and so A(M) is a neighborhood of M. Therefore M is asymptotically stable.

**References**


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