FUBINI PRODUCTS OF LIMINAL $C^*$-ALGEBRAS
WITH HAUSDORFF SPECTRA

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1. Introduction.

Let $A$ and $B$ be $C^*$-algebras and $A \otimes B$ denote the minimal tensor product of $A$ and $B$. After Tomiyama [11] introduced the notion of Fubini product $A \otimes_F B$ of $A$ and $B$, it has been proved to be useful to study some pathological properties of minimal tensor products [1, 4, 8, 12, 13]. Tomiyama [11] also proved that if $A$ is subhomogeneous, that is, every irreducible representation of $A$ is finite dimensional with bounded dimension, then $A$ is a $C^*$-algebra with trivial Fubini products, i.e., $A \otimes_F B = A \otimes B$ for all $C^*$-algebra $B$.

By using the techniques of Wassermann [12, 13], Huruya [5] gave an example of $C^*$-algebra whose irreducible representations are all finite-dimensional but which has a nontrivial Fubini product. So, it is only natural to consider the converse of Tomiyama's result. In this vein, the author [9] showed that the converse is true for AF $C^*$-algebras. The purpose of this note is to prove the converse of the Tomiyama's result for the class of liminal $C^*$-algebras with Hausdorff spectra.

The structures of liminal $C^*$-algebras with Hausdorff spectra are well understood through the theory of continuous fields of $C^*$-algebras, and we follow Dixmier's book [3] for the related notations and terminologies. Also, $K(H)$ (respectively $K(H)$) denotes the $C^*$-algebra of all bounded linear (respectively compact) operators on the separable Hilbert space $H$, throughout this note.

In Section 2, we review definitions and some basic properties of Fubini products. Section 3 is devoted to an extension of Kirchberg's result [7] on exact $C^*$-algebras, which will be useful for the proof of the main theorem in the last section.
2. Preliminaries.

To begin with, let us recall the definition of Fubini product. Let \( A, B, C \) and \( D \) be \( C^* \)-algebras with \( A \subseteq C \) and \( B \subseteq D \). For \( \phi \in C^* \) (respectively \( \phi \in D^* \)), there exists a unique bounded linear map (called the slice map) \( R_\phi : C \otimes D \to D \) (respectively \( L_\phi : C \otimes D \to C \)) such that \( R_\phi (c \otimes d) = \phi (c) d \) (respectively \( L_\phi (c \otimes d) = \phi (d) c \)) for \( c \in C \) and \( d \in D \). The Fubini product \( F(A, B, C \otimes D) \) of \( A \) and \( B \) with respect to \( C \otimes D \) is defined by

\[
F(A, B, C \otimes D) = \{ x \in C \otimes D ; R_\phi (x) \in B, L_\phi (x) \in A \ \\
\text{for all } \phi \in C^*, \phi \in D^* \}.
\]

Let \( A \) and \( B \) be fixed \( C^* \)-algebras. Although the Fubini products \( F(A, B, C \otimes D) \) of \( A \) and \( B \) depend on \( C \otimes D \), they are all isomorphic and constitute the largest among them if \( C^* \)-algebras \( C \) and \( D \) are injective [5]. We denote by \( A \otimes_f B \) any one of these isomorphic Fubini products of \( A \) and \( B \).

Let \( M_n \) denote the \( C^* \)-algebra of all \( n \times n \) matrices over complex field, and \( M \) (respectively \( M_0 \)) be the \( f^* \)-sum (respectively \( c_0 \)-sum) of the family \( \{ M_n \} \) of \( C^* \)-algebras. That is,

\[
M = \{ (x_n) \in \prod_{n=1}^{\infty} M_n ; \sup \| x_n \| < \infty \} \\
M_0 = \{ (x_n) \in \prod_{n=1}^{\infty} M_n ; \lim \| x_n \| = 0 \}.
\]

Then \( M_0 \) is just the example of Huruya [5] mentioned in the introduction:

\[
\mathcal{B}(\mathcal{H}) \otimes M_0 \subseteq F(\mathcal{B}(\mathcal{H}), M_0, \mathcal{B}(\mathcal{H}) \otimes M).
\]

The following two lemmas deal with \( C^* \)-subalgebras and \( C^* \)-quotients for Fubini products in some special cases, which are useful for our main theorem.

**Lemma 2.1.** Let \( A \) and \( C \) be nuclear \( C^* \)-algebras with \( A \subseteq C \). If \( C \) is a \( C^* \)-algebra with trivial Fubini products, then so is \( A \).

**Proof.** See [9, Lemma 3.1].

**Lemma 2.2.** Let \( \alpha : D \to E \) be a surjective \(*\)-homomorphism between \( C^* \)-algebras and \( B \) be a \( C^* \)-subalgebra of \( D \). If \( A \) is a \( C^* \)-algebra then we have

\[
(1_A \otimes \alpha) (F(A, B, A \otimes D)) \subseteq F(A, \alpha(B), A \otimes E).
\]
**Fubini Products of liminal C*-algebras with Hausdorff spectra**

Furthermore, if \( \text{Ker} \alpha \subseteq B \) then the equality holds in (2.1).

**Proof.** If \( z \in F(A, B, A \otimes D) \) then \( R_\phi((1_A \otimes \alpha)(z)) = \alpha(R_\phi(z)) \in \alpha(B) \) for all \( \phi \in A^* \). So, we have \( (1_A \otimes \alpha)(z) \in F(A, \alpha(B), A \otimes E) \). For the converse, let \( w \in F(A, \alpha(B), A \otimes E) \). Then we can find \( z \in A \otimes D \) such that \( (1_A \otimes \alpha)(z) = w \). Now,
\[
\alpha(R_\phi(z)) = R_\phi((1_A \otimes \alpha)(z)) = R_\phi(w) \in \alpha(B)
\]
for all \( \phi \in A^* \). So, \( R_\phi(z) \in B + \text{Ker} \alpha = B \) for all \( \phi \in A^* \), and \( z \in F(A, B, A \otimes D) \). Hence, \( w = (1_A \otimes \alpha)(z) \in (1_A \otimes \alpha)(F(A, B, A \otimes D)) \).

3. **Exact C*-algebras.**

Now, we recall that a C*-algebra \( A \) is C*-exact if
\[
A \otimes J = F(A, J, A \otimes B)
\]
for every C*-algebra \( B \) and its two-sided norm-closed ideal \( J \). Note that the Fubini product \( F(A, J, A \otimes B) \) of the right side is just the kernel of the *-homomorphism \( A \otimes B \rightarrow A \otimes (B/J) \) in general. In the literature [2,7], one can find several conditions which are equivalent to C*-exactness in terms of Fubini product. Especially, Kirchberg [7, Theorem 1.1] showed that if
\[
A \otimes \mathcal{K}(\mathcal{H}) = F(A, \mathcal{K}(\mathcal{H}), A \otimes \mathcal{K}(\mathcal{H}))
\]
then \( A \) is C*-exact.

Let \( E_n \) be a fixed C*-algebra which admits a finite-dimensional irreducible representation with dimension larger than or equal to \( n \), for \( n = 1, 2, ..., \), and denote by \( E \) (respectively \( E_0 \)) the \( l^\infty \)-sum (respectively \( c_0 \)-sum) of \( \{E_n ; n=1, 2, ..., \} \), throughout this section. We show that \( \mathcal{K}(\mathcal{H}) \) (respectively \( \mathcal{K}(\mathcal{H}) \)) can be replaced by \( E \) (respectively \( E_0 \)) in the above mentioned Kirchberg's result.

**Lemma 3.1.** Let \( A \) be a C*-algebra and \( x \in A \otimes M_n \). Then for any \( \varepsilon > 0 \), there exist a completely positive contraction \( W : M_n \rightarrow E_n \) such that
\[
\| (1_A \otimes W)(x) \| > \| x \| - \varepsilon.
\]

**Proof.** By [6, Lemma 2] (see also [10, Lemma 2.7]), there exist completely positive contractions \( W : M_n \rightarrow E_n \) and \( V ; E_n \rightarrow M_n \) such that
\[
\| V W - id \|_{cb} < \frac{\varepsilon}{\| x \|}.
\]

Then, we have
\[ \| (1_A \otimes V) (1_A \otimes W) (x) - x \| < \varepsilon \]

and it follows that
\[ \|x\| - \varepsilon < \| (1_A \otimes V) (1_A \otimes W) (x) \| \leq \| (1_A \otimes W) (x) \|. \]

**Lemma 3.2.** Let \( A \) and \( B \) be \( C^* \)-algebras and \( s \in A \otimes B \). Then, we have
\[
(3.1) \quad \|s\| = \sup \{ \| (1_A \otimes V) (s) \| ; V \text{ is a completely positive } \]
\[
\text{contraction from } B \text{ to } E_n, n=1, 2, \ldots \}. \]

**Proof.** Let \( \varepsilon > 0 \) be given. For a completely positive contraction \( V : B \to M_n \) we can choose, by Lemma 3.1, a completely positive contraction \( W : M \to E_n \) such that
\[ \| (1_A \otimes W) (1_A \otimes V) (s) \| \geq \| (1_A \otimes V) (s) \| - \varepsilon. \]
Now, \( WV : B \to E_n \) is a completely positive contraction and we have
\[ \| (1_A \otimes WV) (s) \| \geq \| (1_A \otimes V) (s) \| - \varepsilon. \]
Hence, the right side of (3.1) is larger than or equal to
\[ \|s\| = \sup \{ \| (1_A \otimes V) (s) \| ; V \text{ is a completely positive contraction } \]
\[ \text{from } B \text{ to } M_n, n=1, 2, \ldots \}, \]
which is the equality proved in [7, Lemma 2.3].

In order to follow the proof of [7, Theorem 1.1], we adopt all notations in [7] such as \( m(B) \), \( c_0(b) \) and \( p_n \). Let \( S \) be the set of all completely positive contractions \( V \) from \( m(B) \) into \( E \) such that \( V(c_0(B)) \subseteq E_0 \).

**Theorem 3.3.** Let \( A \) be a \( C^* \)-algebra. Then, \( A \) is \( C^* \)-exact if and only if
\[ F(A, E_0, A \otimes E) = A \otimes E_0. \]

**Proof.** The arguments of [7] go well except that of [7, Lemma 2.3], for which we will give the following substitute:

**Lemma 3.4.** Let \( t \in A \otimes m(B) \). Then we have

i) \( t \in A \otimes c_0(B) \) if and only if \( (1_A \otimes V) (t) \in A \otimes E_0 \) for every \( V \in S \).

ii) \( t \in F(A, c_0(B), A \otimes m(B)) \) if and only if \( (1_A \otimes V) (t) \in F(A, E_0, A \otimes E) \) for every \( V \in S \).

**Proof.** Assume that \( t \in A \otimes m(B) \setminus A \otimes c_0(B) \). Then, there exists a strictly increasing sequence \( \{ v(n) ; n=1, 2, \ldots \} \) such that \( \| (1_A \otimes p_{v(n)}) (t) \| > 2 \varepsilon, \) for all \( n=1, 2, \ldots \). So, by Lemma 3.2, there exists completely positive contractions \( V_n : B \to E_{v(n)} \) such that
for all \( n = 1, 2, \ldots \). If we define \( V : m(B) \to E \) by
\[
V(b_1, b_2, \ldots) = (0, \ldots, 0, V_1(b_1), 0, \ldots, V_2(b_2), \ldots),
\]
where each \( V_n(b_\nu(n)) \) is in the \( \nu(n) \)-th position, then \( V \in S \).

Let \( \pi_n : E \to E_n \) denote the projection onto the \( n \)-th component. Then, every \( s \in A \otimes E_0 \) satisfies \( \lim_{n \to \infty} \| (1_A \otimes \pi_n) (s) \| = 0 \). But, we have
\[
\| (1_A \otimes \pi_{\nu(n)}) (1_A \otimes V) (t) \| = \| (1_A \otimes V_n) (1_A \otimes \phi_{\nu(n)}) (t) \| \geq \varepsilon
\]
for all \( n = 1, 2, \ldots \), which shows that \( (1_A \otimes V) (t) \in A \otimes E \setminus A \otimes E_0 \). The remaining statements are easy.

4. Main Result.

Throughout the remainder of this note, let \( A \) denote a liminal C*-algebra with Hausdorff spectrum \( T \), and \( \mathcal{A} = \{(A(t))_{t \in T}, \Gamma\} \) be the continuous field of C*-algebras defined by \( A \). Then, by [3, Theorem 10.5.4], we have
\[
A \cong \{ x \in \Gamma ; \lim_{t \to \infty} \| x(t) \| = 0 \}.
\]

**Theorem 4.1.** Let \( A \) be a liminal C*-algebra with Hausdorff spectrum. Then, \( A \) is a C*-algebra with trivial Fubini products if and only if \( A \) is subhomogeneous.

**Proof.** It suffices to prove the necessity. To do this, we assume that \( A \) is not subhomogeneous and show that \( A \) has a nontrivial Fubini product. We consider the following two cases:

Case I: \( A \) has an infinite-dimensional irreducible representation.

In this case, we have \( A(t_0) = \mathcal{K}(\mathcal{H}) \) for some \( t_0 \in T \). Define \( \tilde{A}(t) = A(t) \) for \( t \neq t_0 \) and \( \tilde{A}(t_0) = \mathcal{K}(\mathcal{H}) \). Let \( A \) be the set of \( y \in \prod_{t \in T} \tilde{A}(t) \) such that \( t \mapsto \| y(t) \| \) is continuous on \( T \) and \( y \) coincides to some \( x \in \Gamma \) on the set \( T \setminus \{ t_0 \} \). Then, there exists a unique subset \( \tilde{\Gamma} \) of \( \prod_{t \in T} \tilde{A}(t) \) containing \( A \) such that \( \mathcal{A} = (\tilde{A}(t), \tilde{\Gamma}) \) is a continuous field of C*-algebras on \( T \). Put
\[
\tilde{A} = \{ x \in \tilde{\Gamma} ; \lim_{t \to \infty} \| x(t) \| = 0 \}.
\]
Then, \( A \) is naturally embedded in \( \tilde{A} \). We define \( \alpha : \tilde{A} \to \mathcal{K}(\mathcal{H}) \) by \( \alpha(y) = y(t_0) \) for \( y \in \tilde{A} \). Then \( \alpha \) is surjective and \( \alpha(A) = \mathcal{K}(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H}) \). Furthermore \( \text{Ker } \alpha \subseteq A \). So, we have
by Lemma 2.2. But, we know that the Fubini product of the right side is nontrivial [13, Theorem 8], which implies that $A$ also has a nontrivial Fubini product in the left side.

Case II: Every irreducible representation of $A$ is finite-dimensional.

We can write $T = \bigcup_{n=1}^{\infty} T(n)$, where $T(n)$ is the set of all irreducible representations whose dimensions are less than or equal to $n$. Then $T(n)$ is closed [3, Proposition 3.6.3] and $T$ is of second category because $T$ is locally compact [3, Corollary 3.3.8]. Now, it is easy to see that there exist a strictly increasing sequence $\{n_i\}$ of natural numbers and sequences $\{V_i\}$ of open subsets in $T$ such that $V_i \subset T(n_i) \setminus T(n_i - 1)$.

Put $S = \bigcup_{i=1}^{\infty} V_i$, and let $\mathcal{A}|_{V_i} = (A(t)_{t \in V_i}, \Gamma_{V_i})$ and $\mathcal{A}|_S = (A(t)_{t \in S}, \Gamma_S)$ be the continuous fields induced on $V_i (i=1, 2, \ldots)$ and $S$, respectively. Define

$$E_i = \{ x \in \Gamma_{V_i} : \lim_{t \to \infty} ||x(t)|| = 0 \}$$

$$E_0 = \{ x \in \Gamma_S : \lim_{t \to \infty} ||x(t)|| = 0 \}.$$ 

Then, since each $E_i$ is $n_i$-homogeneous and $E_0$ is the $c_0$-sum of $\{E_i : i=1, 2, \ldots\}$, it follows that $E_0$ has a nontrivial Fubini product by Theorem 3.3. Also, there exists an embedding $E_0 \hookrightarrow A$ defined by $x|\rightarrow \tilde{x}$, where

$$\tilde{x}(t) = \begin{cases} x(t), & \text{for } t \in S \\ 0, & \text{for } t \notin S. \end{cases}$$

Now, both $A$ and $E_0$ are nuclear $C^*$-algebras and we see that $A$ is a $C^*$-algebra with a nontrivial Fubini product by Lemma 2.1. This completes the proof.

Added in proof: Professor Huruya and the author showed that Theorem 4.1 holds for general $C^*$-algebras in their paper “Fubini products of $C^*$-algebras and applications to $C^*$-exactness”.
References


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