ON THE NUMBER OF PRIMES BETWEEN TWO
FUNCTION-VALUES
f(n) AND f(n+1)

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1. Introduction

Let \( \pi(x) \) be the number of primes \( p \) not exceeding \( x \), then the well-known Bertrand's postulate can be written as

\[
\pi(2n) - \pi(n) \geq 1 \quad \text{for all } n \in \mathbb{N} = \{1, 2, 3, \ldots \}.
\]

This is first proved by Chebychev [1]. Actually, it can be shown that for any fixed real number \( r \in (7/12, 1) \)

\[
(1) \quad \pi(n + n^r) - \pi(n) \sim \frac{n^r}{\ln n} \quad \text{as } n \to \infty.
\]

The purpose of this paper is to investigate the number of primes between two function-values \( f(n) \) and \( f(n+1) \), or more formally, to investigate

\[
\Delta_f(n) = f(n+1) - f(n), \quad n \in \mathbb{N}, \quad n \geq a,
\]

where \( f \) is a positive function defined on an interval \([a, \infty)\). Recently, we obtain Theorem A below via the following heuristic result:

\[
(2) \quad p_{n+1} - p_n = O((\ln p_n)^\beta) \quad \text{(as } n \to \infty) \quad \text{for some } \beta \geq 2,
\]

in which \( p_n \) denotes the \( n \)th prime. (Too [8, Theorem 2].)

**THEOREM A.** Let \( r > 1 \) and the function \( f_r(x) = x^r, x \geq 1 \). Then, under the hypothesis (2), there exists an \( n_0(r) \) such that

\[
(3) \quad \Delta_{f_r}(n) \geq 1 \quad \text{for all } n \geq n_0(r).
\]

For polynomial functions

\[
(4) \quad f_k(x) = x^k, \quad x \geq 1, \quad \text{where } k = 2, 3, \ldots,
\]

Hu and Lin [3] obtained the following asymptotic behavior of \( \Delta_{f_k} \) by an elementary proof.

**THEOREM B.** Let \( f_k \) be a function defined in (4). Then, if the ratio

\[
\Delta_{f_k}(n) / (n^{k-1}/\ln n)
\]

--- 199 ---
tends to a limit as $n \to \infty$, the limit must be 1.

In what follows, we are concerned with the larger class of functions (in comparison to (4))

\[(5) \quad f_{a, \beta}(x) = x^a \ln^\beta x, \quad x \geq 1,\]

where the exponents $\alpha$ and $\beta$ satisfy either (i) $\alpha = 1, \beta \geq 1$ or (ii) $\alpha > 1, \beta \in (-\infty, \infty)$. For the case $\alpha > \frac{12}{5}$, we find the rate of convergence of $\Delta f_{a, \beta}$ (Theorem 1), a more precise estimate than (3); for the rest (in fact, for both cases (i) and (ii)), we obtain a result similar to Theorem B by an elementary proof (Theorem 2).

**Theorem 1.** Let $\alpha > \frac{12}{5}$ and $\beta \in (-\infty, \infty)$, then

\[(6) \quad \Delta f_{a, \beta}(n) \sim n^{a-1} \ln^\beta - 1 n \text{ as } n \to \infty.\]

**Theorem 2.** Assume (i) $\alpha = 1, \beta \geq 1$ or (ii)' $1 < \alpha \leq \frac{12}{5}, \beta \in (-\infty, \infty)$. Then, for $f = f_{a, \beta}$ if the ratio

\[(7) \quad \frac{\Delta f(n)}{(f'(n)/\ln f(n))}\]

tends to a limit as $n \to \infty$, the limit must be 1.

Finally, for general functions $f$, we have two interesting results concerning $\Delta f$ (Theorems 3 and 4). The identities (9) and (11) hold for any function $f_{a, \beta}$ defined in (5) since it satisfies all the conditions of Theorems 3 and 4 with $r \geq 1$ and $a$ sufficiently large (see the proof of Theorem 2). Hence Theorem 4 is an extension of Lemma 1 of Hu and Lin [3] who considered the special case $r = 1$ and the polynomial functions $f_k$ defined in (4).

**Theorem 3.** Let $f$ be a positive function defined on an interval $[a, \infty)$ such that $f'$ is positive and nondecreasing. Assume, in addition, that

\[(8) \quad \sum_{i \in [a]} \frac{(f'(i+1))^2}{f(i)f(i+1)} = o (\ln \ln f(n+1)) \text{ as } n \to \infty.\]

Then

\[(9) \quad \lim_{n \to \infty} \frac{1}{\ln \ln f(n+1)} \sum_{i \in [a]} \frac{\Delta f(i)}{f(i)} = 1.\]

**Theorem 4.** Let $f$ be a positive function defined on an interval $[a, \infty)$ such that $f'$ is positive and nondecreasing. Assume further that for a real
On the number of primes between two function-values $f(n)$ and $f(n+1)$

number $r \geq 1$

\begin{equation}
\sum_{i \in \{x\}} \frac{(f'(i+1))^2 \ln f(i)}{f(i) f(i+1)} = o \left( \ln^2 f(n+1) \right) \quad \text{as } n \to \infty.
\end{equation}

Then

\begin{equation}
\lim_{n \to \infty} \frac{r}{\ln^2 f(n+1)} \sum_{i \in \{x\}} \frac{\Lambda_f(i) \ln f(i)}{f(i)} = 1.
\end{equation}

2. Lemmas

To prove the theorems above, we need the next two lemmas. Lemma 1 is due to Huxley [4]. Lemma 2 is interesting in its own right since it is an extension of the well-known Mertens' first theorem written as

\begin{equation}
\sum_{p \leq x} \ln p = \ln x + O(1) \quad \text{as } x \to \infty.
\end{equation}

(see Mertens [6] or Hardy and Wright [2] or Yaglom and Yaglom [9, p.40]).

**Lemma 1.** Let $\varepsilon > 0$, then as $x \to \infty$

\begin{equation}
\pi(x) - \pi(x-y) \sim \frac{y}{\ln x} \quad \text{if } y \in \left[\frac{x^{1+\varepsilon}}{2}, \frac{x}{2}\right].
\end{equation}

**Lemma 2.** Let $r \geq 1$ be a real number. Then

\begin{equation}
\sum_{p \leq x} \frac{\ln^r p}{p} = \frac{1}{r} \ln^r x + O(\ln^{r-1} x) \quad \text{as } x \to \infty.
\end{equation}

**Proof.** It suffices to prove that (14) holds for $r > 1$ since (14) is exactly (12) when $r=1$. Define

$$M_r(x) = \sum_{p \leq x} \frac{\ln^r p}{p}, \quad x > 1,$$

and set

\begin{equation}
M_1(x) = \ln x + R(x),
\end{equation}

so that, by (12),

\begin{equation}
R(x) = O(1) \quad \text{as } x \to \infty.
\end{equation}

Then

\begin{equation}
M_r(x) = \sum_{2 \leq n \leq x} (\ln^{r-1} n) (M_1(n) - M_1(n-1)).
\end{equation}

Inserting (15) into (17) yields

\begin{equation}
M_r(x) = \sum_{2 \leq n \leq x} (\ln^{r-1} n) \ln \frac{n}{n-1}
\end{equation}
We first estimate $S_2(x)$. Summing by parts,

$$S_2(x) = (\ln^{-1}[x]) R([x]) + \sum_{2 \leq n \leq x-1} R(n) \{\ln^{-1}n - \ln^{-1}(n+1)\}.$$ 

From (16) it follows that as $n \to \infty$

$$R(n) \{\ln^{-1}n - \ln^{-1}(n+1)\} = O(\ln^{-1}n - \ln^{-1}(n+1)),$$

so that as $x \to \infty$, 

$$S_2(x) = (\ln^{-1}[x]) R([x]) + O(\ln^{-1}x) = O(\ln^{-1}x).$$

As for $S_1(x)$ in (18), setting $\ln \frac{n}{n-1} = \frac{1}{n} + E(n)$, we have $E(n) = O(n^{-2})$ as $n \to \infty$, hence

$$S_1(x) = \sum_{2 \leq n \leq x} \frac{\ln^{-1}n}{n} + \sum_{2 \leq n \leq x} E(n) \ln^{-1}n$$

$$= \frac{1}{r} \ln x + O(\ln^{-1}x) \quad \text{as} \quad x \to \infty.$$

Combining (19) and (20), we have proved that

$$M_r(x) = S_1(x) + S_2(x)$$

$$= \frac{1}{r} \ln x + O(\ln^{-1}x) \quad \text{as} \quad x \to \infty,$$

which is the desired result.

3. Proofs of theorems

**Proof of Theorem 1.** For any fixed real numbers $\alpha > \frac{12}{5}$ and $\beta$, let us define

$$x_n = f_{\alpha, \beta}(n), \quad y_n = x_{n+1} - x_n, \quad n \in \mathbb{N}.$$

Then, by the mean-value theorem, there exists $\theta_n \in (0, 1)$ such that for sufficiently large $n$

$$y_n = f_{\alpha, \beta}'(n + \theta_n)$$

$$= \alpha (n + \theta_n) \alpha^{-1} \ln \beta (n + \theta_n) + \beta (n + \theta_n) \alpha^{-1} \ln \beta^{-1} (n + \theta_n)$$

$$= (\alpha \ln (n + \theta_n) + \beta) (n + \theta_n) \alpha^{-1} \ln \beta^{-1} (n + \theta_n)$$

$$\geq \alpha n \alpha^{-1} \ln \beta^{-1} (n + \theta_n) = y_n^*.$$
On the number of primes between two function-values $f(n)$ and $f(n+1)$

Choosing $0<\varepsilon<\frac{5}{12}$, we obtain that for all sufficiently large $n$
\[\frac{n^{2+\varepsilon}}{2^{0.5}} \leq y_n \leq \frac{1}{2} x_{n+1},\]
and hence, by Lemma 1,
\[\Delta_{f,\beta}(n) \sim \frac{y_n}{\ln x_{n+1}} \quad \text{as } n \to \infty,
\]
or equivalently
\[\Delta_{f,\beta}(n) \sim \frac{f'(x)\beta(n)}{\ln f\alpha\beta(n)} \sim n^{a-1} \ln \beta^{-1} n \quad \text{as } n \to \infty.
\]

**Proof of Theorem 2.** To prove this theorem, we may apply either Theorem 3 or Theorem 4. It is seen that $f>0$, $f'>0$ and $f''>0$ on $[a, \infty)$ for some sufficiently large integer $a$. Also, for this $f$ the LHS of (8) is
\[O\left(\sum_{i=a}^{n} i^{-2}\right) = O(1) \quad \text{as } n \to \infty.
\]
Therefore, the function $f$ satisfies all the conditions of Theorem 3 and hence the identity (9). Namely,
\[1 = \lim_{n \to \infty} \frac{1}{\ln f(n+1)} \sum_{i=a}^{n} \frac{1}{i \ln i} \cdot \frac{\Delta_f(i)}{i^{a-1} \ln \beta^{-1} i}.
\]
Now, suppose that the ratio (7) tends to a limit $c$, say, as $n \to \infty$, namely,
\[\lim_{n \to \infty} \frac{\Delta_f(n) \ln f(n)}{f'(n)} = c,
\]
or equivalently
\[\lim_{n \to \infty} \frac{\Delta_f(n)}{n^{a-1} \ln \beta^{-1}} = c.
\]
Then we want to prove $c=1$. Note that
\[\lim_{n \to \infty} \frac{\sum_{i=a}^{n} \frac{1}{i \ln i}}{\ln \ln f(n+1)} \sim \ln \ln n \sim \ln \ln f(n+1) \quad \text{as } n \to \infty.
\]
Combining (22) and (23) yields
\[\lim_{n \to \infty} \frac{1}{\ln \ln f(n+1)} \sum_{i=a}^{n} \frac{1}{i \ln i} \frac{\Delta_f(i)}{i^{a-1} \ln \beta^{-1} i} = c.
\]
Therefore, $c=1$ by (21) and (24). The proof is complete.

**Proof of Theorem 3.** For convenience, denote $I_f(i) = (f(i), f(i+1)]$ and assume without loss of generality that $a$ is a positive integer. At first, the monotonicity of $f$ implies that for all $n \geq a$
Secondly, pay attention to the second summation in (25). From Mertens’ second theorem (see, e.g., Yaglom and Yaglom [9, p. 41]) written as

\[ \sum_{p \leq x} \frac{1}{p} = \ln \ln x + O(1) \quad \text{as } x \to \infty \]

it follows that

\[ \lim_{x \to \infty} \frac{\ln \ln f(n+1)}{x} \sum_{p \in \{\infty, f(n+1)\}} \frac{1}{p} = 1. \]

Finally, in view of (9), (25) and (26) it remains to prove that the difference between two sides of (25) is \( o(\ln \ln f(n+1)) \) as \( n \to \infty \). As expected, this difference is

\[
D = \sum_{i=a}^{n} \frac{A_f(i)}{f(i)f(i+1)}(f(i+1) - f(i))
\]

\[
\leq \sum_{i=a}^{n} \frac{(f(i+1) - f(i))^2}{f(i)f(i+1)}
\]

\[
\leq \sum_{i=a}^{n} \frac{(f'(i+1))^2}{f(i)f(i+1)}
\]

\[= o(\ln \ln f(n+1)) \quad \text{as } n \to \infty, \]

in which the last equality follows from the assumption (8). The proof is complete.

**Proof of Theorem 4.** By the conditions on \( f \), \( \lim f(n) = \infty \), so that \( f(x) > e^x \) for all \( x \geq b \), where \( b \) is some sufficiently large integer greater than \( a \). The rest of the proof is similar to that of Theorem 3. From the monotonicity of the function \( (\ln x)/x \) on \((e^x, \infty)\) it follows that for all \( n \geq b \)

\[
\sum_{i=b}^{n} \sum_{p \in f(i)} \frac{\ln f(i+1)}{f(i+1)} \leq \sum_{i=b}^{n} \sum_{p \in f(i)} \frac{\ln f(p)}{p}
\]

\[
\leq \sum_{i=b}^{n} \sum_{p \in f(i)} \frac{\ln f(i)}{f(i)},
\]

and hence

\[
\sum_{i=b}^{n} \frac{A_f(i) \ln f(i+1)}{f(i+1)}
\]
On the number of primes between two function-values \( f(n) \) and \( f(n+1) \)

\[
\leq \sum_{p \in (f(n), f(n+1))} \frac{\ln p}{p} \leq \sum_{i=b}^{a} \frac{A_f(i) \ln f'(i)}{f(i)}.
\] 

Now, applying (14) to the second summation in (27), we obtain that

\[
\lim_{n \to \infty} \frac{r}{\ln f(n+1)} \sum_{p \in (f(n), f(n+1))} \frac{\ln p}{p} = 1.
\]

Finally, in view of (11), (27) and (28) it remains to prove that the difference between two sides of (27) is \( o(\ln f(n+1)) \) as \( n \to \infty \). As expected, this difference is

\[
D^* = \sum_{i=b}^{a} \frac{A_f(i)}{f(i)f(i+1)} \{f(i+1) \ln f(i) - f(i) \ln f(i+1)\}
\]

\[
\leq \sum_{i=b}^{a} \frac{A_f(i)}{f(i)f(i+1)} (f(i+1) - f(i)) \ln f'(i)
\]

\[
\leq \sum_{i=b}^{a} \frac{(f(i+1) - f(i))^2 \ln f(i)}{f(i)f(i+1)}
\]

\[
\leq \sum_{i=b}^{a} \frac{(f'(i+1))^2 \ln f(i)}{f(i)f(i+1)}
\]

\[
= o(\ln f(n+1)) \quad \text{as} \quad n \to \infty,
\]

in which the last equality follows from the assumption (10). The proof is complete.

4. Remarks

Applying Lemma 1 directly we can prove that (1) holds for \( r \in (\frac{7}{12}, 1] \); but the question, whether (1) holds for \( r \in (0, \frac{7}{12}] \), is still open. It will be worth while to mention that if the Riemann hypothesis is true, then (1) holds for \( r \in (\frac{1}{2}, \frac{7}{12}] \) and (6) holds for \( \alpha > 2 \) and \( \beta \in (-\infty, \infty) \) (see Ingham [5] and Titchmarsh [7, p. 77]).

References

15(1972), 164-170.


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