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A Subgroup $P(X, f, x_0)$ of Fundamental Group in the Fixed Point Theory*

by

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1. Introduction

Let X be a topological space with x_0 as a base point. A homotopy h_t: $X \longrightarrow X$ is called a *cyclic homotopy based at* 1_X, if $h_0 = h_i = 1_X$. For a given self-map f: $X \longrightarrow X$, a homotopy h_t: $X \longrightarrow X$ is called a *cyclic homotopy based at* f if $h_0 = h_1 = f$. If h_t is a cyclic homotopy(based at 1_x or based at f), the path σ: $I \longrightarrow X$ given by $\sigma(t) = h_t(x_0)$ will be called the *trace* of h_t. The trace is obviously a loop.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy at 1_X form a subgroup of the fundamental group $\pi_1(X, x_0)$ which we denote by $G(X, x_0)$.

The set of homotopy classes of those loops which are the trace of some cyclic homotopy at f: $X \longrightarrow X$ form a subgroup of $\pi_1(X, f(x_0))$ which we denote by $T(f, f(x_0))$. $T(f, f(x_0))$ is called the Jiang subgroup of f.

The set of elements of $\pi_1(X, f(x_0))$ which operate trivially on $f_* \pi_n(X, x_0)$, for all $n \geq 1$ form a subgroup which will be denoted as $P(X, f, x_0)$.

In the study of fixed point properties of a continuous self-map f: $X \longrightarrow X$ on a connected compact ANR's(absolute neighborhood retracts), several interesting numbers are associated with f. These are the Reidemeister number of f, denoted by $R(f)$, the Lefschetz number of f, denoted by $L(f)$, and the Nielsen number of f, denoted by $N(f)$. S. Lefschetz showed that if $L(f) \neq 0$ then any map g homotopic to f must have a fixed point on X. Jiang proved that when $T(f, f(x_0)) = \pi_1(X, f(x_0))$ (Jiang condition), every fixed point class has the same index k, and $L(f) = kN(f)$.

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Here we are interested is to find the nice properties when $P(X, f, x_0)=\pi_1(X, f(x_0))$.

2. Preliminaries

We will introduce some terminologies and notations, and will summarize those definitions and results that will be used from here on.

If $A=(a_{ij})$ is an $n\times n$ matrix, then the *trace* of A, denoted $Tr\,(A)$, is defined by

$$TrA=\sum_{i=1}^{n} a_{ij}.$$

If A and B are $n\times n$ matrices, then

$$Tr(AB)=\sum_{i,j} a_{ij}b_{ij}=Tr(BA).$$

If G is a free abelian group with basis $e_1,\cdots,e_n$ and if ϕ: $G\longrightarrow G$ is a homomorphism, we define the *trace* of ϕ to be the number $Tr(A)$, where A is the matrix of ϕ relative to the given basis. This number is independent of the choice of basis, since the matrix of ϕ relative to another basis equals $B^{-1}AB$ for some matrix B, and $Tr(B^{-1}(AB))=Tr(AB)B^{-1})=Tr(A)$.

If $f_\sharp$: $C_p(X)\longrightarrow C_p(X)$ is the induced chain map by a given map f: $X\longrightarrow X$, then since $C_p(X)$ is free abelian, the trace of $f_\sharp$ is defined. We denote it by $Tr(f_\sharp, C_p(X))$. The group $H_p(X)$ is not necessarily free abelian, but if $T_p(X)$ is its torsion subgroup, then the group $H_p(X)/T_p(X)$ is free abelian. Furthermore, f_*: $H_p(X)\longrightarrow H_p(X)$ induces a homomorphism f_*: $H_p(X)/T_p(X)\longrightarrow H_p(X)/T_p(X)$. We denote the trace of f_* by $Tr(f_*, H_p(X)/T_p(X))$.

For a given map f: $X\longrightarrow X$, the number

$$L(f)=\sum(-1)^p\, Tr(f_*, H_p(X)/T_p(X)$$

is called the *Lefschetz number* of f.

We always assume X to be a connected compact ANR(absolute neighborhood rectract) with x_0 as a base point. It is well known that X has a universal covering space. Let p: $\hat{X}\longrightarrow X$ be the universal covering of X. A lifting of a map f: $X\longrightarrow X$ is a map $\hat{f}$: $\hat{X}\longrightarrow\hat{X}$ such that $p\cdot\hat{f}=f\cdot p$. A covering transformation is a homeomorphism r: $\hat{X}\longrightarrow\hat{X}$ such that $p\cdot r=p$, that is, a homeomorphic lifting of the identity map 1_x: $X\longrightarrow X$. The set of all covering transformations form a group, we denote it by

$G(\hat{X},X)$ which is isomorphic to $\pi_1(X,x_0)$ (Sometimes we will write $\pi_1(X)$ rather that $\pi_1(X,x_0)$ for brevity). The set of all fixed points of $f\colon X\longrightarrow X$ we will denote by $\mathrm{Fix}(f)$. Two liftings $\hat{f}$ and $\hat{f}'$ of $f\colon X\longrightarrow X$ are said to be conjugate if there exists $r\in G(\hat{X},X)$ such that $\hat{f}'=r\cdot\hat{f}\ r^{-1}$. The *lifting classes* are the equivalence classes by conjugacy. A lifting class containing $\hat{f}$ is denoted by $[\hat{f}]$, that is,

$$[\hat{f}]=\{r\cdot\hat{f}\cdot r^{-1}\,|\,r\in G(\hat{X},X)\}.$$

The following results are well known:

(i) $\mathrm{Fix}(f)=\bigcup_{\hat{f}} p\ \mathrm{Fix}(\hat{f})$

(ii) $p\ \mathrm{Fix}(\hat{f})=p\ \mathrm{Fix}(\hat{f}')$, if $[\hat{f}]=[\hat{f}']$

(iii) $p\ \mathrm{Fix}(\hat{f})\cap p\ \mathrm{Fix}(\hat{f}')=\phi$, if $[\hat{f}]\neq(\hat{f}')$ ([9], p.5)

The subset $p\ \mathrm{Fix}(\hat{f})$ of $\mathrm{Fix}(\hat{f})$ is called the *fixed point class* of f determined by the lifting class $[\hat{f}]$. Thus the fixed point set $\mathrm{Fix}(f)$ is splited into a disjoint union of fixed point classes. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, denoted by $R(f)$. It is a positive integer or infinity. Here we introduce two results for the fixed point classes:

(1) Every fixed point class of $f\colon X\longrightarrow X$ is an open subset of $\mathrm{Fix}(f)$.

(2) Every map $f\colon X\longrightarrow X$ has only finitely many non-empty fixed point classes, each a compact subset of X([9], p.7).

Now we introduce the definition of the fixed point index of a self-map([2] p.51, and [9], p.11). The fixed point index is an indispensable tool of fixed point theory. It provides an algebraic count of fixed points in an open set. There are many different approaches to the fixed point index, all turn out to be equivalent, hence an axiomatic approach has emerged and existence and uniqueness proved. We will introduce the axiomatic approach.

By a graded free abelian group we mean a graded abelian group $G=\{G_p\}$ $p\in Z$ such that each G_p is free. A *finitary* graded abelian group G is defined by the following condition:

(1) $G=\{G_p\}$ $p\in Z$ is a graded free abelian group

(2) each G_p is finitely generated

(3) G_p is trivial except for a finite number of integer P.

Let C_r denote the collection of all connected spaces X with the property that the graded cohomology group $H^*(X:Q)$ is finitary. Let C be a subcollection of C_P. A triple (X,f,U) will be called *C-admissiblle* if

(1) $X \in C$

(2) $f: X \longrightarrow X$ is a map

(3) U is open in X

(4) there are no fixed points of f on the boundary of U.

Observe that if $X \in C$ and $f: X \longrightarrow X$ is a map, then (X,f,X) and (X,f,ϕ) are C-admissible because condition (4) is vacuously satisfied. The symbol C' will be used to denote the collection of all C-admissible triples.

For C a subcollection of C_P, a *fixed point index* on C is a function $i: C' \longrightarrow Q$ which satisfies the following axioms:

Axiom 1 (Localization). If $(X,f,U) \in C'$ and $g: X \longrightarrow X$ is a map such that

$$g(x)=f(x) \text{ for all } x \in U, \text{ then}$$
$$i(X,f,U)=i(X,g,U).$$

Axiom 2 (Homotopy). For $X \in C$ and $H: X \times I \longrightarrow X$ a homotopy, define $f_t: X \longrightarrow X$ by $f_t(x)=H(x,t)$. If $(X,f,U) \in C'$ for all $t \in I$, then

$$i(X,f_0,U)=i(X,f_1,U).$$

Axiom 3 (Additivity). If $(X,f,U) \in C'$ and $U_1,\cdots,U_s$ is a set of mutually disjoint open subsets of U such that $f(x) \neq x$ for all $x \in (U - \bigcup_{j=1}^{s} U_j)$, then

$$i(X,f,U)=\sum_{j=1}^{s} i(X,f,U_k).$$

Axiom 4 (Normalization). If $X \in C$ and $f: X \longrightarrow X$ is a map, then

$$i(X,f,X)=L(f).$$

Axiom 5 (Commutativity). If $X, Y \in C$ and $f: X \longrightarrow Y$, $g: Y \longrightarrow X$ are maps such that $(X,gf,U) \in C'$, then

$$i(X,gf,U)=i(Y,fg,g^{-1}(U)).$$

Let X be a connected compact ANR, $U \subset X$ an open subset and $f: U \longrightarrow X$ a map. Then the fixed point index $i(X,f,U)$ is also defined ([12], p.80).

Let $f: U \longrightarrow X$ be a map such that Fix(f) is compact. A set of fixed points $S \subset$

Fix(f) is called an *isolated set of fixed points* if S is compact and open in Fix(f), that is, if both S and Fix(f)-S are compact. The index of an isolated set S of fixed points, denoted by $i(X, f, S)$, is defined as follows. Pick a neighborhood $W \subset U$ of S isolating S from other fixed points, that is, such that $S = W \cap \text{Fix}(f)$. Define

$$i(X, f, S) = i(X, f, W).$$

Let F be a fixed point class of $f: X \longrightarrow X$. F is an isolated set of fixed points, since F is an open subset of Fix(f). So, $i(X, f, F)$ is defined. F is *essential* if $i(X, f, F) \neq 0$, *inessential* if $i(X, f, F) = 0$. The number of essential fixed point classes of f is called the *Nielsen number* of f, denoted $N(f)$. We have the following results (by the definitions):

(i) $N(f) \leq R(f)$

(ii) Each essential fixed point class is non-empty.

(iii) $N(f)$ is a non-negative integer, $0 \leq N(f) < \infty$.

(iv) $N(f) \leq$ Cardinal number of Fix(f).

Definition 1. The set of homotopy classes of those loops which are the trace of some cyclic homotopy at 1_X form a subgroup of the fundamental group $\pi_1(X, x_0)$ which we shall denote by $G(X, x_0)$ or simply by $G(X)$, that is,

$$G(X, x_0) = \{[\alpha] \in \pi_1(X, x_0) \mid \text{ there exists a cyclic homotopy } h_t: 1_X \simeq 1_X \text{ such that } h_t(x_0) = \alpha(t)\} \quad ([6], \text{ p. } 840).$$

Definition 2. The another trace subgroup of cyclic homotopies at $f: X \longrightarrow X$, denoted by $T(f, f(x_0))$ or simply by $T(f)$ ($\subset \pi_1(X, f(x_0))$) is defined by

$$T(f, f(x_0)) = \{[\alpha] \in \pi_1(X, f(x_0)) \mid \text{there exists a cyclic homotopy } h_t: f \simeq f \text{ such that } h_t(x_0) = \alpha(t)\}.$$

$T(f, f(x_0))$ is called the Jiang subgroup of f, is indeed a subgroup of $\pi_1(X, f(x_0))$ ([2], p. 97 and [9], p. 31). By the definitions, we know that $G(X, x_0)$ is exactly the same as $T(1_X, x_0)$. $T(1_X, x_0)$ is abbreviated by $T(X)$.

The known results are listed without the proof and we label them as propositions.

Proposition 1. If two maps $f, g: X \longrightarrow X$ are homotopic, then $T(f, f(x_0))$ is isomorphic to $T(g, g(x_0))$ ([1], p. 14).

Any path $\gamma: I \longrightarrow X$ such that $\gamma(0)=x_0$, $\gamma(1)=x_1$ induces an isomorphism

$$\gamma_*: \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$$

defined by $\gamma_*[\alpha]=[\gamma\cdot\alpha\cdot\gamma^{-1}]$, for any $[\alpha]\in\pi_1(X, x_1)$ ([5], p.41). Up to isomorphism, $T(f, f(x_0))$ is independent of the choice of base point x_0. Suppose that X is path-connected. For any two points $x_0, x_1 \in X$, there exists a path $\sigma: I \longrightarrow X$ such that $\sigma(0)=x_0$, $\sigma(1)=x_1$. Now $f\cdot\sigma$ is a path from $f(x_0)$ to $f(x_1)$ which induces the isomorphism

$$(f\cdot\sigma)_*: \pi_1(X, f(x_1)) \longrightarrow \pi_1(X, f(x_0)).$$

Moreove r we have the following result:

Pro position 2. The restriction of $(f\cdot\sigma)_*$ to the subgroup $T(f, f(x_1))$ is the isomorphism from $T(f, f(x_1))$ to $T(f, f(x_0))$ ([1], p.15).

Let X be a connected compact ANR with x_0 as a base point.

Proposition 3. Let $f: X \longrightarrow X$ be any map and x_0 be any point of X. Then there exists a map $\bar{f}: X \longrightarrow X$ such that $\bar{f} \simeq f$ and $\bar{f}(x_0)=x_0$ ([1], p.17).

Remark II. By the above results, we see that any statement, up to isomorphism, about $T(f, f(x_0))$ can be proved by proving the statement for $T(\bar{f}, x_0)$. So in the seguel we will generally assume $f(x_0)=x_0$, and we may suppress the base point by writting $T(f)$ for $T(f, f(x_0))$.

Let $\sigma: I \longrightarrow X$ be any path such that $\sigma(0)=x_0$, $\sigma(1)=x_1$.

Let α be any element of $\pi_n(X, x_1)$ and choose a representative map

$$f: (I^n, \partial I^n) \longrightarrow (X, x_1)$$

for α.

Define a partial homotopy $\phi_t: \partial I^n \longrightarrow X(t\in I)$ of f by taking $\phi_t(\partial I^n)=\sigma(1-t)$ for each $t\in I$.

Since ∂I^n has the AHEP(absolute homotopy extension property) in I^n ([5], p.14, proposition 9.2), the homotopy ϕ_t has an extension $f_t: I^n \longrightarrow X(t\in I)$ such that $f_0=f$. Since f_1 maps ∂I^n into $\sigma(0)=x_0$, it represents an element β of $\pi_n(X, x_0)$. By defining $\sigma_n(\alpha)=\beta$, we have the following result.

For each $n>0$, every path σ: $I \longrightarrow X$ such that $\sigma(0)=x_0$, $\sigma(1)=x_1$, gives in a natural way an isomorphism

$$\sigma_n: \pi_n(X, x_1) \longrightarrow \pi_n(X, x_0),$$

which depends only on the homotopy class of the path σ(relative to end points) If σ is the degenerate path $\sigma(I)=x_0$, then σ_n is the identity automorphism ([7], p. 126).

As an immediate consequence of the above fact, we know that the fundamental group $\pi_1(X, x_0)$ operates on $\pi_n(X, x_0)$, $n \geq 1$, as a group of automorphism.

A multiplicative group H is said to operate (or act) on an additive group G(or, simply H operates on G), if for every $h \in H$ and every $g \in G$, an element $hg \in G$ is defined in such a way that

$$h(g_1+g_2)=hg_1+hg_2, \quad h_2(h_1 g)=(h_2 h_1)g, \quad 1g=g,$$

where g, g_1, $g_2 \in G$, h, h_1, $h_2 \in H$ and $1 \in H$ is the neutral element. We say that H operates (or acts) trivially (or simply) on G if $hg=g$ for every $h \in H$ and every $g \in G$.

Let X be any topological space. In the case of $n=1$, we can see that, for any two elements g and h in $\pi_1(X, x_0)$, h operates on g as follows:

$$hg=hgh^{-1}.$$

Hence, if $\pi_1(X, x_0)$ is abelisn, $hg=g$ for every $h \in \pi_1(X, x_0)$ and every $g \in \pi_1(X, x_0)$, that is, if $\pi_1(X, x_0)$ is abelian, $\pi_1(X, x_0)$ operates trivially on $\pi_1(X, x_0)$ itself.

Definition 3. For a given integer $n>0$, a path-connected space X is said to be *n-simple* if there exists a point $x_0 \in X$ such that $\pi_1(X, x_0)$ operates trivially on $\pi_n(X, x_0)$.

By the above fact, a path-connected space X is 1-simple if and only if $\pi_1(X, x_0)$ is sbelian.

From now on we shall concern ourselves only with connected compact ANRs in this paper. Let X be one such with x_0 as a base point.

Definition 4. The set of elements of $\pi_1(X, x_0)$ which operates trivially on $\pi_n(X, x_0)$ for $n \geq 1$ form a subgroup which will be denoted as $P(X, x_0)$ ([6], p. 843).

The above two definitions lead to the following:
A space X is n-simple for all $n \geq 1$ if and only if $P(X, x_0)=\pi_1(X, x_0)$.

The subgroup of $\pi_1(X, x_0)$ which operates trivially on $\pi_1(X, x_0)$ itself is precisely the center of $\pi_1(X, x_0)$, hereafter denoted by $Z(\pi_1(X))$. Thus we have

$$P(X, x_0) \subseteq Z(\pi_1(X)).$$

By Gottlieb, we have

$$G(X, x_0) \subseteq P(X, x_0) \quad ([6], \text{ p.843}).$$

So we have

$$G(X, x_0) \subseteq P(X, x_0) \subseteq Z(\pi_1(X)) \subseteq \pi_1(X, x_0). \cdots\cdots\cdots\cdots\cdots (A)$$

By Brown, we also have

$$G(X, x_0) \subseteq T(f, x_0) \quad ([2], \text{ p. 101}). \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (B)$$

Definition 5. The set of elements of $\pi_1(X, f(x_0))$ which operate trivially on $f_*\pi_n(X, x_0)$, for all $n \geq 1$ will be denoted as $P(X, f, x_0)$, that is, $P(X, f, x_0) = \{[\alpha] \in \pi_1(X, f(x_0)) \mid [\alpha] \text{ operates trivially on } f_*\pi_n(X, x_0),\ n \geq 1\}$.

Proposition 4. $P(X, f, x_0)$ is a subgroup of $\pi_1(X, f(x_0))$. ([5], Theorem 4).

Since any element of $T(f, f(x_0))$ operates trivially on $f_*\pi_n(X, x_0)$, for $n \geq 1$ ([1], p. 44), we have the following:

$$T(f, f(x_0)) \subseteq P(X, f, x_0). \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (C)$$

In the case of $n=1$, we have the following:

Proposition 5. $P(X, f, x_0) \subseteq Z(f_*\pi_1(X),\ \pi_1(X, f(x_0)))$, where $Z(f_*\pi_1(X),\ \pi_1(X, f(x_0)))$ means the centralizer of $f_*\pi_1(X)$ in $\pi_1(X, f(x_0))$. ([5], Theorem 5)

By the above fact and (C), we have

$$T(f, f(x_0)) \subseteq P(X, f, x_0) \subseteq Z(f_*\pi_1(X),\ \pi_1(X, f(x_0))) \subseteq \pi_1(X, f(x_0)). \cdots (D)$$

Proposition 6. If the two maps $f, g: X \longrightarrow X$ are homotopic, then

$$P(X, f, x_0) \cong P(X, g, x_0) \text{ (isomorphic). ([5], Theorem 6)}$$

For a map $f: X \longrightarrow X$, the Jiang condition (i.e., $T(f, f(x_0)) = \pi_1(X, f(x_0))$) assures us a lot of good properties in dealing with numbers of fixed points of f ([8]).

3. Statement of the main result

What nice properties can we conclude when

$$T(f,f(x_0))\subseteq P(X,f,x_0)=\pi_1(X,f(x_0))?$$

Since by (D), we have that

$$T(f,f(x_0))\subseteq P(X,f,x_0)\subseteq Z(f_*\pi_1(X),\ \pi_1(X,f(x_0))\subseteq\pi_1(X,f(x_0)),$$

if $P(X,f,x_0)=\pi_1(X,f(x_0))$, then we have

$$Z(f_*\pi_1(X),\ \pi_1(X,f(x_0))=\pi_1(X,f(x_0)). \quad\cdots\cdots\cdots\cdots\cdots\cdots(E)$$

(E) is equivalent to that every element $[\alpha]\in\pi_1(X,f(x_0))$ commutes with every element $[f\sigma]\in f_*\pi_1(X)$. And this means $f_*\pi_1(X)\subset Z(\pi_1(X))$ (Center of $\pi_1(X)$).

Thus we have the following result:

Theorem 7. If f: $X\longrightarrow X$ is a map such that $P(X,f,x_0)=\pi_1(X,f(x_0))$, then we have

$$f_*\pi_1(X)\subset Z(\pi_1(X)),$$

that is, $f_*\pi_1(X)$ is abelian.

Definition 6. We say f: $X\longrightarrow X$ is 1-*simple* if the image of $f_*:\pi_1(X,x_0)\longrightarrow\pi_1(X,f(x_0))$ is contained in the center of $\pi_1(X,f(x_0))$. By this definition, we have the following:

Corollary 8. If f: $X\longrightarrow X$ is a map such that $P(X,f,x_0)=\pi_1(X,f(x_0))$, then f is 1-simple.

By *R*. Brwon, we have the following four results, we label them as lemmas, and we will use them as lemmas hereafter.

Lemma 9. Let f: $X\longrightarrow X$ be a map such that $T(f,f(x_0))=\pi_1(X,f(x_0))$, then all the fixed point classes of f have the same index. ([12], p.99, Theorem 4).

Lemma 10. If X is connected aspherical (i.e. $\pi_n(X)=0$ for all $n>1$) and f: $X\longrightarrow X$ is a map, then

$$Z(f_*(\pi_1(X)),\ \pi_1(X,f(x_0))\subseteq T(f,f(x_0))\ \ ([2],\ 102,\ \text{Theorem } 10).$$

By (D), we have $T(f, f(x_0)) = Z(f_* \pi_1(X), \pi_1(X, f(x_0))$.

Lemma 11. Suppose that $f: X \longrightarrow X$ is a map such that $T(f, f(x_0)) = \pi_1(X, f(x_0))$. If $L(f)=0$, then $N(f)=0$. ([2], p. 100, Cor. 5).

Lemma 12. If $f: X \longrightarrow X$ is a map such that $T(f, f(x_0)) = \pi_1(X, f(x_0))$, then $N(f)=R(f)$. ([2], p. 106, Cor. 6).

Theorem 13. Suppose that X is connected aspherical. If $P(X, f, x_0) = \pi_1(X, f(x_0))$, then we have $T(f, f(x_0)) = \pi_1(X, f(x_0))$, that is the Jiang condition is satisfied.

Proof. Since X is connected aspherical, by Lemma 10, we have $T(f, f(x_0)) = Z(f_* \pi_1(X)), \pi_1(X, f(x_0))$. By (E), we have

$$T(f, f(x_0)) = \pi_1(X, f(x_0)). \quad ///$$

If X is connected aspherical and $P(X, f, x_0) = \pi_1(X, f(x_0))$, then, since the Jiang condition is satisfied, the nice properties in the fixed point theory follow:

Corollary 14. Suppose that X is connected aspherical. If $P(X, f, x_0) = \pi_1(X, f(x_0))$, then we have the following:

(1) All the fixed point classes of f have the same index.

(2) If $L(f)=0$, then $N(f)=0$.

(3) $N(f)=R(f)$.

Proof. The proof is immediate from the above Lemmas.

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