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Applications of Convolution Operators to some Classes of Close-to-convex Functions

by

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Abstract

Let $C[C,D]$ and $S^*[C,D]$ denote the classes of functions g, $g(0)=1-g'(0)0=0$, analytic in the unit disc E such that $\frac{(zg'(z))'}{g'(z)}$ and $\frac{zg'(z)}{g(z)}$ are subordinate to $\frac{1+Cz}{1+Dz}$, $z\in E$, respectively. In this paper, the classes $K[A,B;C,D]$ and $C^*[A,B;C,D]$, $-1\leq B<A\leq 1$; $-1\leq D<C\leq 1$, are defined. The functions in these classes are close-to-convex. Using the properties of convolution operators, we deal with some problems for our classes.

1. Introduction

Let S, K, S^* and C denote the classes of analytic functions f: $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are respectively univalent, close-to-convex, starlike (with respect to origin) and convex in the unit disc E. In [3], a new subclass C^* of univalent functions was introduced and studied. A function f belongs to C^* if there exists a convex function g such that, for $z\in E$,

$$\mathrm{Re}\ \frac{(zf'(z))'}{g'(z)}>0.$$

The functions in C^* are called quasi-convex functions and CC^*KS. For more details of C^*, see Noor [4].

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In [2], Janowski introduced the class $P[A,B]$. For A and B, $-1\le B<A\le 1$, a function p, analytic in E with $p(0)=1$ belongs to the class $P[A,B]$ if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$. Also, given C and D, $-1\le D<C\le 1$, $C[C,D]$ and $S^*[C,D]$ denote the classes of functions, analytic in E with $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ such that $\frac{(zf'(z))'}{f'(z)} \in P[C,D]$ and $\frac{zf'(z)}{f(z)} \in P[C,D]$ respectively. For $C=1$, and $D=-1$, we note that $C[1,-1]=C$ and $S^*[1,-1]\equiv S^*$.

Silvia [8] defines the classes $K[A,B;C,D]$ as follows:

Definition 1. 1

A function f: $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, analytic in E, is said to be in the class $K[A,B; C,D]$, $-1\le B<A\le 1$, $-1\le D<C\le 1$, if there exists a $g\in C[C,D]$ such that $\frac{f'(z)}{g'(z)} \in P[A,B]$. It is clear that

$$K[1,\ -1:\ 1,\ -1]\equiv K.$$

and

$$K[A,B;\ C,D]\subset K\subset S.$$

We now define the following:

Definition 1. 2

A function f: $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, analytic in E, is said to be in the class $C^*[A,B; C,D]$, $-1\le B<A\le 1$; $-1\le D<C\le 1$, if there exists a $g\in C[C,D]$ such that $\frac{(zf'(z))'}{g'(z)} \varepsilon P[A,B]$.

Clearly $C^*[1,\ -1;\ 1,\ -1]\equiv C^*$. We also note that $f\in C^*[A,B;\ C,D]$ if, and only if $zf'\in K[A,B;\ C,D]$.

Let f and g be analytic in E with $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and $g(z)=\sum_{n=0}^{\infty} n_n z^n$. Then the convolution (or the Hadamard product) of f and g is defined by

$$(f*g)\ (z)=\sum_{n=0}^{\infty} a_n b_n z^n.$$

Given f analytic in E, we define the convolution operator Γ: $A\longrightarrow A$: by $\Gamma(g)=f*g$, where A is the class of functions analytic in E. By the Hadamard product, we also define

$$D^{\alpha}f(z)=\frac{z}{(1-z)^{1+\alpha}}*f(z) \quad (\alpha\geq 1)$$

for $f\in A$. $D^{\alpha}f(z)$ is called the Ruscheweyh derivative and was introduced by Ruscheweyh in [6].

Let Γ_i, $0\leq i\leq 4$ be the linear operators defined on A by the following:

$$\Gamma_0 f(z)=zf'(z),\ \Gamma_1 f(z)=\frac{f(z)+zf'(z)}{2},$$

$$\Gamma_2 f(z)=\int_0^z \frac{f(\xi)-f(0)}{\xi}\,d\xi,$$

$$\Gamma_3 f(z)=\frac{2}{z}\int_0^z f(\xi)\,d\xi,$$

and

$$\Gamma_4 f(z)=\int_0^z \frac{f(\xi)-f(x\xi)}{\xi-x\xi}\,d\xi,\ |x|\leq 1,\ x\neq 1.$$

We note that each of these operators can be written as a convolution operator given by

$$\Gamma_i f=h_i*f,\ 0\leq i\leq 4,$$

where

$$h_0(z)=\sum_{n=1}^{\infty} nz^n=\frac{z}{(1-z)^2},$$

$$h_1(z)=\sum_{n=1}^{\infty}\frac{n+1}{2}z^n=\frac{z-z^2/2}{(1-z)^2},$$

$$h_2(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=-\log(1-z),$$

$$h_3(z)=\sum_{n=1}^{\infty}\frac{2}{n+1}z^n=\frac{-2[z+\log(1-z)]}{z},$$

$$h_4(z)=\sum_{n=1}^{\infty}\frac{1-x^n}{(1-x)^n}z^n=\frac{1}{1-z}\log\left[\frac{1-xz}{1-z}\right],\ |x|\leq 1,\ x\neq 1.$$

We also observe that the radius of convexity of each of the functions h_i, $0\leq i\leq 4$ is $r_c(h_0)=2-\sqrt{3}$, $r_c(h_1)=\frac{1}{2}$ and $r_c(h_2)=r_c(h_3)=r_c(h_4)=1$.

2. Preliminary Results

We shall need the following lemmas.

Lemma 2.1 [5]

Let ϕ be convex and f be close-to-convex. Then $\phi * f$ is close-to-convex in E.

Lemma 2.2 [7]

Let $g \in S^*[C, D]$. Then, for ϕ convex, $\phi * g \in S^*[C, D]$.

Lemma 2.3 [8]

If ϕ is convex and $f \in K[A, B;\ C, D]$, then $\phi_* f \in K[A, B;\ C, D]$.

Lemma 2.4 [1]

Let ϕ and g be analytic in E with $\phi(0) = g(0) = 0$ and $\phi'(0)\ g'(0) \neq 0$. Suppose that, for each $\gamma(|\gamma| = 1)$ and $\sigma(|\sigma| = 1)$, we have

$$\left[\phi * \left(\frac{1+\gamma\sigma z}{1-\sigma z}\right) g\right](z) \neq 0$$

on $0 < |z| < \Gamma \leq 1$.

Then, for each F analytic in E, the image of $|z| < r$ under $(\phi * Fg)/(\phi * g)$ is a subset of the convex hull of $F(E)$.

3. Main Results

Theorem 3.1

Let $\phi \in S^*$ and $f \in C^*$. Then $\phi * f$ is close-to-convex.

Proof

$$\begin{aligned}(\phi * f)\ (z) &= (z\psi' * f)\ (z),\ \psi \in C^* \\ &= (\psi * zf')\ (z) \\ &= (\psi * F)\ (z),\ F \in K,\end{aligned}$$

and the result follows by using Lemma 2.1.

Theorem 3.2

Let ϕ be convex and $f \in C^*[A, B;\ C, D]$. Then $\phi * f \in C^*[A, B;\ C, D]$.

Proof

Since $f \in C^*[A, B;\ C, D]$, it follows that $F = zf' \in K[A, B;\ C, D]$. Also

$$(\phi^* F)(z) = \phi(^* zf') = z(\phi^* f)' \in K[A, B;\ C, D]$$

by Lemma 2.3. Hence $\phi^* f \in C^*[A, B;\ C, D]$.

From the definitions of $h_i, 0 \leq i \leq 4$, and theorem 3.2, we have:

Theorem 3.3

Let $f \in C^*[A, B;\ C, D]$. Then $\Gamma_i f = h_i^* f \in C^*[A, B;\ C, D]$ upto $r_c(h_i)$ for each i, $0 \leq i \leq 4$.

We now prove the folloiwng result for the class $C^*[A, B;\ CD]$, and give its application to the generalized hypergeometric functions.

Theorem 3.4

Let $f \in C^*[A, B;\ C, D]$. Then

$$I_c(f(z)) = F_c(z) = \frac{1+c}{z^c} \int_0^z t^{c-1} f(t) dt, \ \text{Re } c > 0$$

is also in $C^*[A, B; C, D]$.

Proof:

Let

$$h_c(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \ \text{Re } c > 0$$

The function $h_c(z)$ is convex, see [6]. Now

$$F_c(z) = (h_c^* f)\ (z)$$

and the result follows immediately from theorem 3.2.

Corollary

Let the generalized hypergeometric function $z\ {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots, \beta_q;\ z)$ with $p \leq q+1$ be in the class $C^*[A, B;\ C, D]$. Then the function

$$z\ {}_{p+k}F_{q+k}(\alpha_1, \cdots, \alpha_p;\ c_1+1, \cdots, c_k+1;\ \beta_1, \cdots, \beta_q,\ c_1+1, \cdots, c_k+2;\ z)$$

is also in the class $C^*[A, B;\ C, D]$, where $c_j \leq 0$ $(j = 1, \cdots, k)$

Proof:

The generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$
\begin{aligned}
{}_pF_q(z) &= {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots, \beta_q;\ z) \\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \quad (p \leq q+1)
\end{aligned}
$$

wlehere

$$\alpha_j (j=1, \cdots, p) \text{ and } \beta_j (j=1, \cdots, q)$$

are complx numbers with $\beta_j \neq 0, -1, -2, \cdots, j=1, 2, \cdots, q$, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$
(\lambda)_n = \begin{cases} 1 & \text{if } n=0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in N = \{1, 2, \cdots\}. \end{cases}
$$

Now

$$
\begin{aligned}
I_c(z\ {}_pF_q(z)) &= \frac{c+1}{z^c} \int_0^z t^{c-1} \{t\ {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots, \beta_q, t)\ dt\} \\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \cdot \frac{c+1}{n+c+1} \frac{z^{n+1}}{n!} \\
&= z\ {}_{p+1}F_{q+1}(\alpha_1, \cdots, \alpha_p, c+1;\ \beta_1, \cdots, \beta_q, c+2;\ z)
\end{aligned}
$$

which, in view of thorem 3.4, belongs to the class $C^*[A, B; C, D]$. Using this method along with theorem 3.4 repeatedly we obtain the required result.

We shall now prove some results related with Ruscheweyyh derivatives.

Theorem 3.5

Let $f \in S^*[A, B]$ and let $D^\alpha f(z) \neq 0$ $(0 < |z| < 1)$ for $\alpha \geqq -1$. Then $D^\alpha f \in S^*[A, B]$.

Proof

$$
\begin{aligned}
\frac{z(D^\alpha f(z))'}{D^\alpha f(z)} &= \frac{D^\alpha (zf'(z))}{D^\alpha f(z)} \\
&= \left[\frac{z}{(1-z)^{1+\alpha}} * zf'(z)\right] \Big/ \frac{z}{(1-z)^{1+\alpha}} * f(z)
\end{aligned}
$$

We take $\gamma = -1$, $\phi(z) = \frac{z}{(1-z)^{1+\alpha}}$, $g(z) = f(z)$ and $F(z) = \frac{zf'(z)}{f(z)}$ in Lemma 2.4 and obtain the required result.

The relationship between the classes $C[A, B]$ and $S^*[A, B]$ gives us immediately the following:

Theorem 3.6

Let $f\in C[A,B]$ and satisfy the condition $D^{\alpha}(zf'(z))\neq 0$ $(0<|z|<1)$ for $\alpha\geq -1$. Then $D^{\alpha}f\in C[A,B]$.

We shall now prove the similar results for the classes $K[A,B;\ C,D]$ and $C^*[A,B;\ C,D]$.

Theorem 3.7

Let $f\in K[A,B;\ C,D]$ with respect to a function $g\in S^*[C,D]$. Let $D^{\alpha}g(z)\neq 0$ $(0<|z|<1)$ for $\alpha\geq -1$. Then $D^{\alpha}f\in K[A,B;\ C,D]$.

Proof

$$\frac{z(D^{\alpha}f(z))'}{D^{\alpha}g(z)}=\frac{D^{\alpha}(zf'(z))}{D^{\alpha}g(z)}$$
$$=\left[\frac{z}{(1-z)^{1+\alpha}}*\frac{zf'(z)}{g(z)}\cdot g(z)\right]\Big/\left[\frac{z}{(1-z)^{1+\alpha}}*g(z)\right]$$

Now applying Lemma 2.4 with $\gamma=-1$, $\phi(z)=\frac{z}{(1-z)^{1+\alpha}}$, $g(z)=g(z)$ and $F(z)=\frac{zf'(z)}{g(z)}$ we obtain the required result.

Using theorem 3.7 along with the fact that $f\in C^*[A,B;\ C,D]$ if and only if $zf'\in K[A,B;\ C,D[$, we have the following.

Theorem 3.8

Let $f\in C^*[A,B;\ C,D]$ with respect to $g\varepsilon C[C,D]$ with the condition $D^{\alpha}(zg'(z))\neq 0$ $(0<|z|<1)$ for $\alpha\geq -1$. Then $D^{\alpha}f\in C^*[A,B;\ C,D]$.

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