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Some Properties of Koszul Complexes

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1. Introduction

The best known of some free complexes is the Koszul complex. This remarkable complex has been generalized by J.A. Eagon and D.G. Northcott ([8]) and by D.A. Buchsbaum and D.S. Rim([3], [4] and [6]). Here we shall follow very closed the united treatment of J.A. Eagon and D.G. Northcott ([8]).

The purpose of this paper is to find some properties of the generalized Koszul complex associated with matrices and to apply to the concept of grade, which has been developed in the seminar performed during the last semester.

In details the contents of this thesis is described as follows; In section 2 we will describe basic properties of the Koszul complex which will be used in to help understand section 4. In section 3 we deal with the minimal resolution over a noetherian local ring, and in section 4, which is the main part of this thesis, we will prove several Theorems.

If $H_{r-s+1}(E^A)\neq 0$ then there exists a non-zero element e of E such that $(A)e=0$ ([8]), but the inverse does not hold. Under suitable conditions we will prove that the inverse holds(Theorem 4.2). Finally, in the noetherian local ring R we will prove in the Theorem 4.5 that, if $gr((A):R)=r-s+1$ then the complex $R^A\longrightarrow R/(A)$ is a minimal resolution of $R/(A)$ and the projective dimension of $R/(A)$ is $r-s+1$, where A is an $s\times r$-matrix$(s\leq r)$ with entries in the maximal ideal, (A) is an ideal of R generated by s-minors of the matrix A and E is an R-module.

As usual R denotes a commutative ring with a non-zero identity element.

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2. Preliminaries

Let $x_1, x_2, \cdots, x_n$ be elements of R.

A complex $K(x_1, x_2, \cdots, x_n)$ is defined as follows;

$$K_0(x_1, \cdots, x_n) = R$$
$$K_r(x_1, \cdots, x_n) = \bigoplus_{1 \le i_1 < \cdots < i_r \le n} Re_{i_1 \cdots i_r} \quad (r = 1, 2, \cdots, n)$$

where $Re_{i_1} \cdots_{i_r}$ is R-free generated by $e_{i_1} \cdots_{i_r}$.

$$d\colon K_r(x_1, \cdots, x_n) \longrightarrow K_{r-1}(x_1, \cdots, x_n)$$

is defined by

$$d(e_{i_1 \cdots i_r}) = \sum_{j=1}^{r} (-1)^{j+1} x_j e_{i_1 \cdots \hat{i}_j \cdots i_r},$$

where $\hat{i}_j$ indicates that i_j is omitted there.

Then it is easy to prove that $d^2 = 0$. Thus

$$K(x_1, \cdots, x_n): 0 \longrightarrow K_n(x_1, \cdots, x_n) \xrightarrow{d} K_{n-1}(x_1, \cdots, x_n) \longrightarrow \cdots \longrightarrow$$
$$K_1(x_1, \cdots, x_n) \xrightarrow{d} K_0(x_1, \cdots, x_n) \longrightarrow 0$$

is a complex which is called a *Koszul complex*. For $x \in R$

$$K(x): \cdots \longrightarrow 0 \longrightarrow K_1(x) \xrightarrow{x} R \longrightarrow 0 \longrightarrow \cdots$$

Let

$$C: \cdots \longrightarrow C_u \longrightarrow C_{u-1} \longrightarrow \longrightarrow \cdots C_1 \longrightarrow C_0 \longrightarrow 0$$

be a chain complex of R-modules, then

$$(C \otimes_R K(x))_u = C_u \otimes_R K_0(x) \oplus C_{u-1} \otimes_R K_1(x)$$
$$\cong C_u \oplus C_{u-1},$$

and we get a complex

$$C \otimes_R K(x): \cdots \longrightarrow C_u \oplus C_{u-1} \longrightarrow C_{u-1} \oplus C_{u-2} \longrightarrow \cdots \longrightarrow C_1 \oplus C_0 \longrightarrow C_0 \longrightarrow 0$$

where the differential d is defined by

$$d(\xi, \eta) = (d\xi + (-1)^{u-1} x\eta,\ d\eta)$$

for each $(\xi, \eta) \in C_u \oplus C_{u-1}$.

We define the complex C' by $C_u' = C_{u-1}$ and $C_0' = 0$, we have the exact sequence

$$0 \longrightarrow C \longrightarrow C \otimes_R K(x) \longrightarrow C' \longrightarrow 0 \qquad (*)_1$$

of complexes.

Therefore we obtain the long exact sequence

$$\cdots \longrightarrow H_u(C) \longrightarrow H_u(C \otimes_R K(x)) \longrightarrow H_u(C') \xrightarrow{(-1)^{u-1}x} H_{u-1}(C) \longrightarrow \cdots$$

Lemma 2.1. Under the above situation $xH_u(C \otimes_R K(x)) = 0$ for all $u \geq 0$.

Proof: For $(\xi, \eta) \in (C \otimes_R K(x))_u$ we assume $d(\xi, \eta) = 0$. Thus $d\eta = 0$ and $d\xi + (-1)^{u-1} x\eta = 0$. Take $(0, (-1)^u \xi) \in (C \otimes K(x))_{u+1}$ then

$$d(0, (-1)^u \xi) = x(\xi, \eta) \in d(C \otimes_R K(x))_{u+1}.$$

This implies that $xH_u(C \otimes_R K(x)) = 0$ for all $u \geq 0$. ■

For an R-module M we shall put

$$K(\underline{x} : M) = K(x_1, \cdots, x_n : M) = K(x_1, \cdots, x_n) \otimes_R M$$

and $H_u(\underline{x} : M) = H_u(x_1, \cdots, x_n : M) = H_u K(x_1, \cdots, x_n : M)$ for all $u \geq 0$.
By $(x_1, \cdots, x_n)$ we mean an ideal of R generated by $x_1, \cdots, x_n$.

Proposition 2.2. For all $u \geq 0$ we have $(x_1, \cdots, x_n)\ H_u(x_1, \cdots, x_n : M) = 0$.

Proof: For any fixed element x_i of $\{x_1, \cdots, x_n\}$, consider two complexes

$$K(x_i) : 0 \longrightarrow K_1(x_i) \longrightarrow R \longrightarrow 0$$

and

$$C = K(x_1, \cdots, \hat{x}_i, \cdots, x_n : M) : 0 \longrightarrow K_{n-1}(x_1, \cdots, \hat{x}_i, \cdots, x_n : M) \longrightarrow K_{n-2}(x_1, \cdots, \hat{x}_i, \cdots, x_n : M) \longrightarrow \cdots \longrightarrow K_1(x_1, \cdots, \hat{x}_i, \cdots, x_n : M) \longrightarrow M \longrightarrow 0.$$

Note that if we permute $x_1, x_2, \cdots, x_n$ in any manner the Koszul complex is unchanged.

By Lemma 2.1, for all $u \geq 0$

$$0 = x_i H_u(C \otimes K(x_i)) = x_i H_u(x_1, \cdots, x_i, \cdots, x_n : M).$$

Therefore $(x_1, x_2, \cdots, x_n)\ H_u(x_1, \cdots, x_n : M) = 0$ for all $u \geq 0$. ■

Lemma 2.3. Let M be an R-module and let $x_1, \cdots, x_n$ be an R-sequence on M. Then

$$H_u(x_1, \cdots, x_n : M) = 0 \text{ if } u > 0$$
$$H_0(x_1, \cdots, x_n : M) = M/(x_1, \cdots, x_n)M.$$

Proof: We shall do this proof by induction on n.
When $n=1$, set $x=x_1$ the sequence

$$K(\underset{\sim}{x} : M) : 0 \longrightarrow \underset{\underset{M}{\wr\wr}}{K_1(x:M)} \xrightarrow{x} \underset{\underset{M}{\wr\wr}}{K_0(x:M)} \longrightarrow 0$$

is exact.

By our hypothesis, since x is a regular element of M it follows that

$$H_1(x:M) = 0 \text{ and } H_0(x:M) = M/xM.$$

Hence the assertion holds for $n=1$.

In general, we assume that for $r<n$ our assertion is true on r.
Putting $C = K(x_1, \cdots, x_{n-1} : M)$ and $K(x) = K(x_n)$ in the formula $(*)_1$ we obtain an exact sequence

$$0 \longrightarrow K_u(x_1, \cdots, x_{n-1} : M) \longrightarrow K_u(x_1, \cdots, x_n : M) \longrightarrow K_{u-1}(x_1, \cdots, x_{n-1} : M) \longrightarrow 0$$

for each u. Therefore, it induces a long exact sequence

$$\cdots \longrightarrow H_u(x_1, \cdots, x_{n-1} : M) \longrightarrow H_u(x_1, \cdots, x_n : M)$$
$$\longrightarrow H_{u-1}(x_1, \cdots, x_{n-1} : M) \longrightarrow H_{u-1}(x_1, \cdots, x_{n-1} : M) \longrightarrow \cdots.$$

By induction hypothesis we have $H_u(x_1, \cdots, x_n : M) = 0$ for each $u \geq 2$. When $u=1$, since

$$0 \longrightarrow H_1(x_1, \cdots, x_n : M) \longrightarrow M/(x_1, \cdots, x_{n-1})M \xrightarrow{x_n} M/(x_1, \cdots, x_{n-1})M \longrightarrow \cdots$$

is exact and x_n is a regular element of $M/(x_1, \cdots, x_{n-1})M$, we have $H_1(x_1, \cdots, x_n : M) = 0$.
Therefore $H_u(x_1, \cdots, x_n : M) = 0$ for all $u \geq 1$.
It is clear that $H_0(x_1, \cdots, x_n : M) = M/(x)M$. ■

Note that $H_n(x_1, \cdots, x_n : E) = 0 :_E (x_1, x_2, \cdots, x_n)$ and we know that, if x_n is not a zero-divisor on E $H_u(x_1, \cdots, x_n : E) = H_u(x_1, \cdots, x_{n-1} : E/x_nE)$ for each u ([14]).

Lemma 2.4. Let R be a commutative ring, E a noetherian R-module and $x_1, \cdots, x_n$ elements in the Jacobson radical of R. If $H_1(x_1, \cdots, x_n : E) = 0$ then $H_u(x_1, \cdots, x_n E) = 0$ for all $u \neq 0$.

Proof: This will be accomplished by using induction on n. The assertion is trivial when $n=1$.

Assume that the assertion has been established when the number of elements involved is only $n-1$. For the long exact sequence

$$\cdots \longrightarrow H_u(x_1, \cdots, x_{n-1} : E) \xrightarrow{(-1)^u x_n} H_u(x_1, \cdots, x_{n-1} : E)$$
$$\longrightarrow H_u(x_1, \cdots, x_n : E) \longrightarrow H_{u-1}(x_1, \cdots, x_{n-1} : E) \longrightarrow \cdots$$

which is obtained by $(*)_1$, if $H_u(x_1, \cdots, x_i : E) = 0$ then $H_u(x_1, \cdots, x_{i-1} : E) = 0$ as well. By repeat this process we see that $H_1(x_1, \cdots, x_j : E) = 0$ for all $j = 1, 2, \cdots, n$. $0 = H_1(x_1 : E) = (0 :_E x_1)$ means that x_1 is not a zero-divisor on E. Put $\bar{E} = E/x_1E$. Then

$$H_1(x_1, \cdots, x_n : E) \cong H_1(x_2, \cdots, x_n, x_1 : E) \cong H_1(x_2, \cdots, x_n : E/x_1E).$$

Thus $H_1(x_2, \cdots, x_n : \bar{E}) = 0$ and therefore, by induction hypothesis, $H_u(x_2, \cdots, x_n : \bar{E}) = 0$ for all $u \neq 0$. Since $H_u(x_2, \cdots, x_n : \bar{E}) \cong H_u(x_1, x_2, \cdots, x_n : E)$, the Lemma is now complete. ■

Consider an $s \times r$-matrix

$$A = \begin{Vmatrix} a_{11} \; a_{12} \cdots a_{1r} \\ \cdots\cdots\cdots\cdots \\ a_{s1} \; a_{s2} \cdots a_{sr} \end{Vmatrix} \quad (s \leq r)$$

with coefficients in the ring R.

Let the exterior algebra generated by $X_1, \cdots, X_r$ be denoted by K, that is,

$$K_0 = R$$
$$K_1 = RX_1 \oplus RX_2 \oplus \cdots \oplus RX_r$$
$$K_2 = \bigoplus_{1 \leq i < j \leq r} RX_iX_j$$
$$\vdots$$
$$K_r = RX_1 \cdots X_r.$$

Define $\Delta_k : K_u \longrightarrow K_{u-1}$ $(1 \leq u \leq r,\ k = 1, 2, \cdots, s)$ by

$$\Delta_k(X_{i_1} \cdots X_{i_u}) = \sum_{p=1}^{u} (-1)^{p+1} a_{ki_p} X_{i_1} \cdots \hat{X}_{i_p} \cdots X_{i_u}$$

where $1\leq i_1<\cdots<i_u\leq r$ and $\hat{X}_{i_p}$ means that X_{i_p} is omitted. It is clear that

$$\Delta_h\Delta_k+\Delta_k\Delta_h=0$$

for all h and k with $h\neq k$.

Let Φ_q be the R-module generated by $Y_1^{v_1}\cdots Y_s^{v_s}(v_i\geq 0,\ v_1+\cdots+v_s=q\leq r-s)$ in the polynomial ring $R[Y_1,\cdots,Y_s]$. Define the R-module R^A_{q+1} such that

$$R^A_{q+1}=K_{s+q}\otimes_R\Phi_q$$

where $0\leq q\leq r-s$. Then R^A_{q+1} is a finitely generated free R-module, because K_{s+q} and Φ_q are finite free R-modules. We define $R^A_0=R$. If $q>r-s$ then $K_{s+q}=0$. Thus $R^A_{r-s+i}=0$ for all $i\geq 2$. Hence we have a complex

$$R^A:\ 0\longrightarrow R^A_{r-s+1}\xrightarrow{d}R^A_{r-s}\longrightarrow\cdots\longrightarrow R^A_1\longrightarrow R^A_0\longrightarrow 0$$

where d is defined as follows

(i) $q\geq 1$: For each generator $X_{i_1}\cdots X_{i_{s+q}}\otimes Y_1^{v_1}\cdots Y_s^{v_s}(1\leq i_1<\cdots<i_{s+q}\leq r,\ v_1+\cdots+v_s=q)$ of R^A_{q+1}.

$$d(X_{i_1}\cdots X_{i_{s+q}}\otimes_R Y_1^{v_1}\cdots Y_s^{v_s})=\sum_j{}^*\Delta_j(X_{i_1}\cdots X_{i_{s+q}})\otimes Y_1^{v_1}\cdots Y_j^{v_j-1}\cdots Y_s^{v_s},$$

where $\sum_j^*$ the summation for j such that $v_j>0$.

(ii) $q=0$: R^A_1 is the free R-module generated by $\{X_{i_1}\cdots X_{i_s}\otimes 1\mid 1\leq i_1<\cdots<i_s\leq r\}$,

$$d(X_{i_1}\cdots X_{i_s}\otimes 1)=\begin{vmatrix} a_{1i_1}\ a_{1i_2}\cdots a_{1i_s} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{si_1}\ a_{si_2}\cdots a_{si_s}\end{vmatrix}.$$

Thus $d^2=0$. Indeed for $q=v_1+v_2+\cdots+v_s\geq 2$

$$\begin{aligned} d^2(X_{i_1}\cdots X_{i_{s+q}}\otimes Y_1^{v_1}\cdots Y_s^{v_s}) &=\sum_{h\neq k}{}^*(\Delta_h\Delta_k+\Delta_k\Delta_h)(X_{i_1}\cdots X_{i_{s+q}})\otimes \\ & Y_1^{v_1}\cdots Y_h^{v_h-1}\cdots Y_k^{v_k-1}\cdots Y_s^{v_s}+\sum_l{}^{**}\Delta_l\Delta_l(X_{i_1}\cdots X_{i_{s+q}})\otimes Y_1^{v_1}\cdots Y_l^{v_l-2}\cdots Y_s^{v_s} \\ &=0\end{aligned}$$

where $\sum_l^{**}$ is the summation for l such that $v_l\geq 2$. And for each $k=1,2,\cdots,s$

$$d^2(X_{i_1}\cdots X_{i_{s+1}}\otimes Y_k)=\begin{vmatrix} a_{ki_1}\,a_{ki_2}\cdots a_{ki_{s+1}} \\ a_{1i_1}\,a_{1i_2}\cdots a_{1i_{s+1}} \\ \cdots\cdots\cdots\cdots \\ a_{si_1}\,a_{si_2}\cdots a_{si_{s+1}} \end{vmatrix}=0.$$

Therefore

$$R^A:0\longrightarrow R^A_{r-s+1}\longrightarrow R^A_{r-s}\longrightarrow\cdots\longrightarrow R^A_1\longrightarrow R^A_0=R\longrightarrow 0$$

is a complex which is called a *generalized Koszul complex.*

Let E be an R-module, then

$$E^A=R^A\otimes_R E:0\longrightarrow R^A_{r-s+1}\otimes_R E\longrightarrow\cdots\cdots\longrightarrow R^A_1\otimes_R E\longrightarrow E\longrightarrow 0$$

is also a complex, we shall use these notations in §4.

Let (A) be the ideal generated by the subdeterminants of A of order s. Then it is clear that $H_0(E^A)=E/(A)E$.

3. Minimal resolutions

Throughout this section $(R,\mathfrak{M},k)$ denotes a noetherian local ring. For R-modules L and M, an R-module homomorphism

$$f:L\longrightarrow M$$

is said to be ***minimal*** if

$$f\otimes 1_k=\bar{f}:L\otimes_R k=\bar{L}\longrightarrow M\otimes_R k=\bar{M}$$

is an isomorphism, or equivalently, if f is surjective and Ker $f\subset\mathfrak{M}L$.

Let M be a finitely generated R-module, and let

$$F.:\cdots\longrightarrow F_n\longrightarrow F_{n-1}\longrightarrow\cdots\longrightarrow F_1\longrightarrow F_0\longrightarrow M\longrightarrow 0$$

be a free resolution of M. If

$$d_i:F_i\longrightarrow \mathrm{Ker}(d_{i-1})\ (i\geq 1)\ \text{and}\ d_0:F_0\longrightarrow M$$

are minimal, then this free resolution $F.$ is called a ***minimal resolution*** of M. If $F.$ is a minimal resolution of M then

$$\bar{F}.=F.\otimes_R k:\cdots\longrightarrow \bar{F}_n\xrightarrow{\bar{d}_n}\bar{F}_{n-1}\longrightarrow\cdots\longrightarrow\bar{F}_1\xrightarrow{\bar{d}_1}\bar{F}_0$$

has trivial differential (*i.e.*, $\bar{d}_i=0$ for all $i\geq 1$), and

$$\bar{d}_0 : F_0 \longrightarrow M$$

is an isomorphism. Therefore

$$\mathrm{Tor}_i^R(M,k)=H_i(F.\otimes_R k)\cong F_i$$

for $i\geq 1$ and

$$\mathrm{Tor}_0^R(M,k)\cong M.$$

We have to note that each F_i is a finitely generated free R-module for $i\geq 0$ and so

$$\mathrm{rank}\ F_i=\mathrm{rank}_k\ \mathrm{Tor}_i^R(M,k)<\infty$$

Proposition 3.1. Let $(R,\mathfrak{M},k)$ be a noetherian local ring, and let $x_1,\cdots,x_n$ be an R-sequence on R. Then the Koszul complex $K(x_1,\cdots,x_n): 0\longrightarrow K_n(x_1,\cdots,x_n)\longrightarrow K_{n-1}(x_1,\cdots,x_n)\longrightarrow\cdots\longrightarrow K_0(x_1,\cdots,x_n)=R\overset{\varepsilon}{\longrightarrow}R/(x_1,\cdots,x_n)\longrightarrow 0$ is a minimal resolution of $R/(x_1,\cdots,x_n)$.

Proof: By Lemma 2.3 it follows that

$$H_u(\underline{x}:R)=0 \text{ if } u>0 \text{ and } H_0(\underline{x}:R)=R/(x_1,\cdots,x_n).$$

This implies that the Koszul complex $K(x_1,\cdots,x_n)$ is a free resolution of the R-module $R/(x_1,\cdots,x_n)$.

Since $x_1,\cdots,x_n\in\mathfrak{M}$ and $k=R/\mathfrak{M}$, it follows that

$$\bar{d}_u: K_u(x_1,\cdots,x_n)\otimes k\longrightarrow K_{u-1}(x_1,\cdots,x_n)\otimes k$$

is trivial. Since $(x_1,\cdots,x_n)\subset\mathfrak{M}$, the map

$$\bar{\varepsilon}: R\otimes k=k\longrightarrow R/(x_1,\cdots,x_n)\otimes k=k$$

is an isomorphism. Therefore $K(x_1,\cdots,x_n)$ is a minimal resolution of $R/(x_1,\cdots,x_n)$. ■

Proposition 3.2. Let $(R,\mathfrak{M},k)$ be a noetherian local ring. M be a finitely generated R-module. Then the following hold; (1) There is a minimal resolution $L.\longrightarrow M$ of M. (2) For an arbitrary free resolution $F.\longrightarrow M$ of M there exists an acyclic complex $W.$ such that $F.=L.\oplus W.$.

Proof: (1) Let M be a finitely generated R-module. Let $\{w_1, \cdots, w_r\}$ be a minimal base of M, *i.e.*, $M=Rw_1+\cdots+Rw_r$ and $M\otimes_R k$ is an r-dimensional vector space over k. Let L_0 be a free R-module with basis $\{e_1, \cdots, e_r\}$, *i.e.*,

$$L_0=Re_1\oplus\cdots\oplus Re_r \ (Re_i\cong R \text{ as } R\text{-modules}).$$

Then there exists an R-homomorphism d_0: $L_0\longrightarrow M$ such that $d_0(e_i)=w_i$.

Then it follows that $L_0\otimes_R k\cong M\otimes_R k$. We have to note that $K_1=\text{Ker } d_0$ is a finitely generated R-module. Let $\{w_1', \cdots, w_t'\}$ be a minimal base of K_1. Put

$$L_1=Re'_1\oplus\cdots\oplus Re'_t \ \ (Re'_i\cong R \text{ as } R\text{-modules})$$

We have there exists an R-homomorphism d_1: $L_1\longrightarrow K_1$ such that $d_1(e_i')=w_i'$, then it follows that $L_1\otimes k\cong K_1\otimes k$. Repeating this way we get a minimal resolution of M:

$$L.: \cdots\longrightarrow L_n\longrightarrow L_{n-1}\longrightarrow\cdots\longrightarrow L_1\longrightarrow L_0\longrightarrow M\longrightarrow 0.$$

(2) Let $F.\longrightarrow M$ be a free resolution of a finitely generated R-module M.

Since any two minimal resolutions of M are isomorphic ([13]), we may assume that the free resolution $F.\longrightarrow M$ is not minimal and so there must be some i such that the matrix $\|a_{ij}\|$ defining $\Phi_i : F_i\longrightarrow F_{i-1}$ does not have coefficients in $\mathfrak{M}$. Since R is local, this means that some a_{ij} is unit. But now a change of bases for F_i and F_{i-1} will transform $\|a_{ij}\|$ to $\left\| \begin{matrix} 1 & 0\cdots\cdots 0 \\ 0 & \\ \vdots & \|a'_{ij}\| \\ 0 & \end{matrix} \right\|$.

This means that we have a commutative diagram

$$\begin{array}{ccc} F_i & \xrightarrow{\Phi_i} & F_{i-1} \\ \wr\| & & \wr\| \\ R\oplus F'_i & \xrightarrow{1\oplus\|a'_{ij}\|} & R\oplus F'_{i-1} \end{array}$$

and the complex $F.$ is the direct sum of

$$\cdots\longrightarrow F_{i+1}\longrightarrow F'_i\longrightarrow F'_{i-1}\longrightarrow F_{i-2}\longrightarrow\cdots$$

and $$\cdots\longrightarrow 0\longrightarrow R\xrightarrow{1} R\longrightarrow 0\longrightarrow\cdots$$

This process can be continued until we are left with a minimal free resolution $L.$

of M and the sum of pieces of the form $\cdots\longrightarrow 0\longrightarrow R\longrightarrow R\longrightarrow\cdots$ in various degrees. Putting these latter pieces together gives $W.$, which is an acyclic complex of free modules. ■

4. Generalized Koszul complexes

For a ring R we shall consider an $s\times r$-matrix

$$A=\left\|\begin{matrix} a_{11}\cdots\cdots a_{1r} \\ \cdots\cdots\cdots\cdots \\ a_{s1}\cdots\cdots a_{sr}\end{matrix}\right\| \quad (s\leq r,\ a_{ij}\in R).$$

Let us put an $s\times(r-1)$-matrix S and an $(s-1)\times(t-1)$-matrix T such that

$$S=\left\|\begin{matrix} a_{11}\cdots a_{1\,r-1} \\ \cdots\cdots\cdots\cdots \\ a_{s1}\cdots a_{s\,r-1}\end{matrix}\right\| \quad\text{and}\quad T=\left\|\begin{matrix} a_{21}\cdots\cdots a_{2\,r-1} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{s1}\cdots\cdots a_{s\,r-1}\end{matrix}\right\|$$

As in §2, we then have three complexes;

$$R^A:\ 0\longrightarrow R^A_{r-s+1}\longrightarrow R^A_{r-s}\longrightarrow\cdots\cdots\longrightarrow R^A_1\longrightarrow R\longrightarrow 0,$$

$$R^S:\ 0\longrightarrow R^S_{r-s}\longrightarrow R^S_{r-s-1}\longrightarrow\cdots\cdots\longrightarrow R^S_1\longrightarrow R\longrightarrow 0,$$

$$R^T:\ 0\longrightarrow R^T_{r-s+1}\longrightarrow R^T_{r-s}\longrightarrow\cdots\cdots\longrightarrow R^T_1\longrightarrow R\longrightarrow 0.$$

There is an R-homomorphism

$$u_{q+1}:R^S_{q+1}\longrightarrow R^T_{q+1}\quad (q=0,1,\cdots,r-s)$$

which is defined as follows: Recall that

$$R^S_{q+1}=\bigoplus_{1\leq i_1<\cdots<i_{s+q}\leq r-1} RX_{i_1}\cdots X_{i_{s+q}}\otimes Y_1^{\nu_1}\cdots Y_s^{\nu_s}$$

where $\nu_1+\cdots+\nu_s=q$. It will be convenient to set such that

$$R^T_{q+1}=\bigoplus_{1\leq j_1<\cdots<j_{s+q-1}\leq r-1} RU_{j_1}\cdots U_{j_{s+q-1}}\otimes V_2^{\eta_2}\cdots V_s^{\eta_s},$$

where $\eta_2+\cdots+\eta_s=q$.

We define such that

(i) $q=r-s$: $u_{q+1}=0$

(ii) $0\leq q\leq r-s-1$;

$$u_{q+1}(X_{i_1}\cdots X_{i_{s+q}}\otimes Y_1^{v_1}\cdots Y_s^{v_s})$$
$$=\begin{cases}\sum_{p=1}^{s+q}(-1)^{p+1}a_{1i_p}U_{i_1}\cdots\hat{U}_{i_p}\cdots U_{i_{s+q}}\otimes V_2^{v_2}\cdots V_s^{v_s} & \text{if } v_1=0\\ 0 \quad \text{if } v_1\geq 1\end{cases}$$

Lemma 4.1. Under the above situation

$$d_{q+1}u_{q+1}+u_q d_{q+1}=0 \quad (q=1,2,\cdots,r-s).$$

Proof: In a term $X_{i_1}\cdots X_{i_{s+q}}\otimes Y_1^{v_1}\cdots Y_s^{v_s}$ if $v_1\geq 2$ then $d_{q+1}u_{q+1}=0=u_q d_{q+1}$. If $v_1=1$ then $d_{q+1}u_{q+1}=0$ and also $u_q d_{q+1}=0$ because that, for example,

$$u_q d_{q+1}(X_{i_1}\cdots X_{i_{s+q}}\otimes Y_1 Y_2^{v_2}\cdots Y_s^{v_s})$$
$$=\Big[a_{1i_1}\Big(\sum_{p=2}^{s+q}(-1)^p a_{1i_p}U_{i_2}U_{i_3}\cdots\hat{U}_{i_p}\cdots U_{i_{s+q}}\Big)$$
$$-a_{1i_2}\Big(a_{1i_1}U_{i_3}\cdots U_{i_{s+q}}+\sum_{p=3}^{s+q}(-1)^p U_{i_1}U_{i_3}\cdots\hat{U}_{i_p}\cdots U_{i_{s+q}}\Big)$$
$$+\cdots+(-1)^{s+q-1}a_{1i_{s+q}}\Big(\sum_{p=1}^{s+q-1}(-1)^{p+1}a_{1i_p}U_{i_1}\cdots\hat{U}_{i_p}\cdots U_{i_{s+q-1}}\Big)\Big]$$
$$\otimes V_2^{v_2}\cdots V_s^{v_s}$$
$$=0.$$

When $v_1=0$, by a straightout calculation as above it follow that $d_{q+1}u_{q+1}+u_q d_{q+1}=0$. ■

Let us put

$$R_0^{S,T}=0,\quad R_1^{S,T}=R_1^T \quad\text{and}\quad R_{q+1}^{S,T}=R_{q+1}^T\oplus R_q^S$$

for $q=1,\cdots,r-s$ and define the complex

$$R^{S,T}:0\longrightarrow R_{r-s+1}^{S,T}\xrightarrow{\delta}R_{r-s}^{S,T}\xrightarrow{\delta}\cdots\longrightarrow R_2^{S,T}\xrightarrow{\delta}R_1^T\longrightarrow 0$$

where δ is defined by

$$\delta(x_q^T,\ x_{q-1}^S)=\begin{cases}(dx_q^T+ux_{q-1}^S,\ dx_{q-1}^S) & \text{if } q>2\\ dx_2^T+x_1^S & \text{if } q=2,\end{cases}$$

for each $(x_q^T, x_{q-1}^S) \in R_q^{S,T} = R_q^T \oplus R_{q-1}^S$. It follows from lemma 4.1 that $\partial^2 = 0$ so that $R^{S,T}$ really is a complex.

Let us define an R-module homomorphism

$$\varphi_{q+1} : R_{q+1}^A \longrightarrow R_{q+1}^{S,T}$$

as follows. For each $X_{i_1} \cdots X_{i_{s+q}} \otimes Y_1^{v_1} \cdots Y_s^{v_s} \in R_{q+1}^A$ with $1 \leq i_1 < \cdots < i_{s+q} \leq r$

$$\varphi_{q+1}(X_{i_1} \cdots X_{i_{s+q}} \otimes Y_1^{v_1} \cdots Y_s^{v_s})$$
$$= \begin{cases} 0 & \text{if } i_{s+q} \leqq r-1 \\ X_{i_1} \cdots X_{i_{s+q-1}} \otimes Y_1^{v_1-1} Y_2^{v_2} \cdots Y_s^{v_s} & \text{if } i_{s+q} = r \text{ and } v_1 \geq 1 \\ U_{i_1} \cdots U_{i_{s+q-1}} \otimes V_2^{v_2} \cdots V_s^{v_s} & \text{if } i_{s+q} = r \text{ and } v_1 = 0. \end{cases}$$

Thus $\varphi : R^A \longrightarrow R^{S,T}$ is a chain map ([8]), and by the definition of the map φ we have the exact sequence

$$0 \longrightarrow R^S \longrightarrow R^A \overset{\varphi}{\longrightarrow} R^{S,T} \longrightarrow 0$$

of complexes, since R_q^S is a direct summand of R_q^A, each of the exact sequences

$$0 \longrightarrow R_q^S \longrightarrow R_q^A \longrightarrow R_q^{S,T} \longrightarrow 0$$

splits.

For an R-module E, the above split exact sequences induce an exact sequence

$$0 \longrightarrow E^S \longrightarrow E^A \longrightarrow E^{S,T} \longrightarrow 0$$

of complexes, where $E^{S,T} = R^{S,T} \otimes_R E$. This gives rise to the exact homology sequence

$$\cdots \longrightarrow H_q(E^S) \longrightarrow H_q(E^A) \longrightarrow H_q(E^{S,T}) \longrightarrow H_{q-1}(E^S) \longrightarrow \cdots \quad (**)_1.$$

Let (T) be the ideal of R which is generated by subdeterminants with order $s-1$ in the matrix T.

Theorem 4.2. Assume that the following conditions hold

(i) $H_{r-s}(E^S) = 0$

(ii) There exists a non-zero element e of E such that $(T)e = 0$ if $H_{r-s+1}(E^T) \neq 0$

(iii) If $(T)e = 0$ for some a non-zero element e of E then

$$\begin{vmatrix} a_{1i_1}\cdots a_{1i_{s-1}}\ a_{1r} \\ \cdots\cdots\cdots\cdots \\ a_{si_1}\cdots a_{si_{s-1}}\ a_{sr} \end{vmatrix} e=0$$

for all $1\leq i_1<\cdots<i_{s-1}\leq r-1$.

Then there exists a non-zero element $e\in E$ such that $(A)\ e=0$ if and only if $H_{r-s+1}(E^A)\neq 0$.

Proof. Suppose that there exists a non-zero element e of E such that $(A)\ e=0$. Then $H_{r-s+1}(E^A)\neq 0$. This proof is found in [8] without any conditions. Conversely, suppose that $H_{r-s+1}(E^A)\neq 0$. By condition (i) there is no any element $\eta\in E^S_{r-s}-0$ with $d\eta=0$. This implies that for $(\xi,\eta)\in E^{S,T}_{r-s+1}$

$$\delta(\xi,\eta)=0 \text{ if and only if } d\xi=0 \text{ and } \eta=0.$$

Hence we have $H_{r-s+1}(E^A)=H_{r-s+1}(E^{S,T})=H_{r-s+1}(E^T)$ by the exact sequence $(**)_1$ above. By condition (ii) if $H_{r-s+1}(E^A)\neq 0$ then there exists an element $e(\neq 0)\in E$ such that $(T)\ e=0$. Since

$$\begin{vmatrix} a_{1i_1}\cdots a_{1i_s} \\ \cdots\cdots\cdots \\ a_{si_1}\cdots a_{si_s} \end{vmatrix} = a_{1i_1}\begin{vmatrix} a_{2i_2}\cdots a_{2i_s} \\ \cdots\cdots\cdots \\ a_{si_2}\cdots a_{si_s} \end{vmatrix} +\cdots+(-1)^{s+1}a_{1i_s}\begin{vmatrix} a_{2i_1}\cdots a_{2i_{s-1}} \\ \cdots\cdots\cdots \\ a_{si_1}\cdots a_{si_{s-1}} \end{vmatrix}$$

By condition (iii), $(A)\ e=0$. ∎

From now on, we assume that R is a noetherian ring. For a finitely generated R-module E and an ideal I of R with $IE\neq E$, a sequence $u_1,\cdots u_n\in I$ is said to be an *R-sequence* on E *in* I if the following two conditions are satisfied

(i) u_i is a regular element of $E/(u_1,\cdots,u_{i-1})E$ for $1\leq i\leq n$

(ii) $(u_1,\cdots,u_n)\ E\neq E$.

If there is not any regular element of $E/(u_1,\cdots,u_n)E$ in I then $u_1,\cdots,u_n$ is said to be maximal. Since any two maximal R-sequences on E in I have the same number of elements ([10]), we shall denote this number by $gr\ (I: E)$.

For a finitely generated R-module E and an $s\times r$-matrix $A=\|a_{ij}\|$, the following has been proved ([8]);

$$gr((A): E)+q=r-s+1, \quad (**)_2$$

where q is the largest value of m such $H_m(E^A)\neq 0$.

This property gives the following theorem:

Theorem 4.3. Let R be a noetherian ring. Let $0\longrightarrow E'\longrightarrow E\longrightarrow E''\longrightarrow 0$ be an exact sequence of finitely generated R-modules and I an ideal of R with $IE'\neq E'$, $IE\neq E$ and $IE''\neq E''$, then

(1) If $gr(I:E)>gr(I:E'')$ then $gr(I:E')=gr(I:E'')+1$

(2) If $gr(I:E)=gr(I:E'')$ then $gr(I:E')\geq gr(I:E)$

(3) If $gr(I:E)<gr(I:E'')$ then $gr(I:E')=gr(I:E)$.

Proof. Let $I=(x_1,\cdots,x_r)$. The exact sequence $0\longrightarrow E'\longrightarrow E\longrightarrow E''\longrightarrow 0$ gives rise to the exact sequence $0\longrightarrow K(x_1,\cdots,x_r:E')\longrightarrow K(x_1,\cdots,x_r:E)\longrightarrow K(x_1,\cdots,x_r:E'')\longrightarrow 0$ of complexes. Then we obtain an exact homology sequence

$$\cdots\longrightarrow H_n(\underline{x}:E')\longrightarrow H_n(\underline{x}:E)\longrightarrow H_n(\underline{x}:E'')\longrightarrow H_{n-1}(\underline{x}:E')$$
$$\longrightarrow\cdots\longrightarrow H_0(\underline{x}:E)\longrightarrow H_0(\underline{x}:E'')\longrightarrow 0 \qquad (**)_3$$

(1) put $gr(I:E'')=s$ then, from the exactness of $(**)_3$ and the property $(**)_2$ $H_t(\underline{x}:E'')=0$ for all $t>r-s$. Since $gr(I:E)>s$ then $H_t(E)=0$ for all $t\geq r-s$ and $H_{r-s}(\underline{x}:E'')\neq 0$. Hence $H_t(\underline{x}:E')=0$ for all $t>r-s$ and $H_{r-s-1}(\underline{x}:E')\neq 0$. Therefore $gr(I:E')=s+1$.

Similarly, we can prove (2) and (3). ∎

Proposition 4.4. Let R be a noetherian local ring and $x_1,\cdots,x_r$ be elements which generate its maximal ideal $\mathfrak{M}$, then the followings are equivalent;

(1) $\dim R=r$

(2) R is regular

(3) $H_1(\underline{x}:R)=0$.

Proof. Clearly (1)$\Longleftrightarrow$(2).

(2)$\Longrightarrow$(3) Since R is regular then $gr(\mathfrak{M}:R)=\dim R=r$. Therefore $H_1(x:R)=0$.

(3)$\Longrightarrow$(1) By lemma 2.4 $H_1(\underline{x}:R)=0$ implies $H_u(\underline{x}:R)=0$ for all $u\neq 0$. $(**)_2$ induces $gr(\mathfrak{M}:R)=r$, then we obtain $\mathrm{ht}(\mathfrak{M})=r$ from the principal ideal theorem and $\mathrm{ht}(I)\geq gr(I:R)$ for any ideal I of R. Therefore $\dim R=r$. ∎

Theorem 4.5. Let R be a noetherian local ring. If $gr((A):R)=r-s+1$ and $a_{ij}\in\mathfrak{M}$ for all i,j then

$$R^A: 0\longrightarrow R^A_{r-s+1}\longrightarrow\cdots\longrightarrow R^A_0\longrightarrow R/(A)\longrightarrow 0 \quad (**)_4$$

is a minimal resolution over $R/(A)$. Moreover the projective dimension of $R/(A)$ is $r-s+1$.

Proof. From $(**)_2$ we know $H_m(E^A)=0$ for all $m>0$ then $(**)_4$ is a free resolution over $R/(A)$, since R_i^A is free for all i. By the definition of d_i, we have Ker $d_i \subset \mathfrak{M} R_i^A$ (Note that $d_{i+1}(R_{i+1}^A)=$ Ker d_i and $a_{ij} \in \mathfrak{M}$). Therefore $(**)_4$ is a minimal resolution over $R/(A)$ ($R_0^A=R$ and $(A) \subset \mathfrak{M}$ implies that $R \otimes_R k \simeq k \simeq R/(A) \otimes_R k$).

Let $P. \longrightarrow R/(A)$ be a projective resolution over $R/(A)$. Since R is local every projective R-module is R-free. Therefore, by Proposition 3.2 there exists an acyclic complex $W.$ such that $P.=R^A \oplus W.$, which means that the projective dimension of $R/(A)$ is $r-s+1$. ■

Corollary Let R be a noetherian local ring and $x_1, \cdots, x_r$ generate the maximal ideal $\mathfrak{M}$. If dim $R=r$ then the projective dimension of $R/\mathfrak{M}$ is r.

Proof. By Lemma 2.4 and Proposition 4.4 $H_u(\underset{\sim}{x}:R)=0$ for all $u \neq 0$. We know $gr(\mathfrak{M}:R)=r$ from $(**)_2$. Therefore by Theorem 4.5 the projective dimension of $R/\mathfrak{M}$ is r. ■

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