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The Functors Γ^+, Γ and Infinite Loop spaces

by

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1. Introduction

The concept of infinite loop spaces was defined by R.J. Milgram in his paper [19] (1966). And then, J.P. May([15], [8], 1969), S.B Priddy([22], 1971), J.P. Stasheff ([25], 1971), P.O. Kirly([14], 1975), J.F. Adams([1], 1978) studied the infinite loop spaces, intensively.

D.W. Anderson([2], 1970), M.G. Barratt([3]—[5]) have studied the Γ^+ and Γ. In this paper, we shall study infinite loop spaces by means of the Γ^+, Γ functors.

In §2, we shall study the properties of the $C_m^n (n \geq m)$, and we define the functors Γ^+ and Γ using the structure of C_m^n.

In §3, we proved the some properties of Γ^+ and Γ-functors. One of these is that for $X, Y \in \mathrm{Obj(Pss)}$ if $f \simeq g$: $X \longrightarrow Y \in \mathrm{Morph(Pss)}$ then $\Gamma^+ f \simeq \Gamma^+ g$ and $\Gamma(f) \simeq \Gamma(g)$ (Theorem 3.4).

§4 is devoted to the main part of this dissertation, the author's goal is to prove that: Let X be an $(n-1)$ connected CW-complex with dimension $\leqq 2n-1$. Then there exists CW-complex W such that $Q(X) \approx Q(SW)$ and $\pi_n^s(X) = \pi_{n-1}^s(W)$ (Theorem 4.4).

Finally, $Q(SX)$ is atomic at 2 and $Q(SY)$ is atomic at 2, where X has the same homotopy type as CP^n $(n=1,2,3,4,5\ldots)$ and Y has the same homotopy type as RP^n $(n=1,2,4,5,6,8,\cdots)$ (Theorem 4.6). Throughout this paper Ω and S denote the based loop and the reduced suspension functors respectively. While writting this paper we referred [7], [9], [16]—[21], [23] for homotopy theory and [11], [13] for K-theory.

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2. Preliminaries

Let X be a based topological space, and let S^iX be the i-th reduced suspension of X. Then, for a given integer $n \geq o$ the Freudenthal suspension homomorphism

$$\varphi_i:\ \pi_{n+i}(S^iX) \longrightarrow \pi_{n+1+i}(S^{i+1}X)$$

is an isomorphism for all $i > n+1$. Therefore, in terms of the direct system $\{\pi_{n+i}(S^iX),\ \varphi_i\}$ we can define the stable homotopy group

$$\pi_n^s(X) = \underset{i}{\underrightarrow{lim}}\ \pi_{n+i}(S^iX).$$

Then, it follows that $\pi_n^s(X) = \pi_{n+i}(S^iX)$ for all $i > n+1$. Since S and Ω are adjoint functors to each other for based spaces X and Y

$$[SX,\ Y] = [X,\ \Omega Y]$$

where $[SX, Y]$ is the homotopy class of continuous functions from SX to Y which preserve base point. Therefore

$$\pi_{n+i}(S^iX) = [S^i \Lambda S^n,\ S^iX] = [S^n,\ \Omega^iS^iX] = \pi_n(\Omega^iS^iX)$$

and thus for all $i > n+1$ we have

$$\pi_n^s(X) = \pi_n(\Omega^iS^iX).$$

Furthermore, in the following

$$\begin{array}{ccc} [S^{n+1}X,\ S^{n+1}X] & = & [S^nX,\ \Omega S^{n+1}X] \\ \Cup & & \Cup \\ [1_{S^{n+1}X}] & \longmapsto & [\psi_{n+1}] \end{array}$$

we have a continuous map $\Omega^n\psi_{n+1}:\ \Omega^nS^nX \longrightarrow \Omega^{n+1}S^{n+1}X$, where $[1_{S^{n+1}X}]$ is the homotopy class of $1_{S^{n+1}X}$. Then we have *a* sequence

$$\Omega SX \xrightarrow[\Omega f_2]{} \Omega^2S^2X \xrightarrow[\Omega^2 f_3]{} \Omega^3S^3X \xrightarrow[\Omega^3 f_4]{} \cdots.$$

Thus we can define

$\underrightarrow{lim}\ \Omega^nS^nX = Q(X)$ where the topology of $Q(X)$ is the topology coinduced by the inclusions $\Omega^nS^nX \longrightarrow Q(X)$ which is an infinite loop space (see the first part of §4).

In what follows, we shall describe the Barratt's Γ or Γ^+-functor ([3]—[5]).

Let S_n be the symmetric group on the set $N=\{1,2,\cdots,n\}$. For a set $M=\{1,2,\cdots,m\}$, we consider a strictly monotonically increasing map $M\longrightarrow N$. The set of those maps will be denoted by C^n_m.

Hence, if $n<m$ then $C^n_m=\phi$ and if $n=m$ then C^n_m consists of only one element. Moreover, each element $\alpha\in C^n_m$ induces a group homomorphism

$$\alpha_*:\ S_m\longrightarrow S_n$$

which is defined as follows. For each $\sigma\in S_m$

$$\begin{aligned} &\alpha_*(\sigma)\ (\alpha(i))=\alpha(\sigma(i)) && \forall i\in\{1,\cdots,m\}\\ &\alpha_*(\sigma)\ (j)=j && \forall j\in N-\alpha(M), \end{aligned}$$

then α_* is a group homomorphism. In fact, for $\sigma_1,\sigma_2\in S_m$

$$\begin{aligned} \alpha_*(\sigma_1\sigma_2)\ (\alpha(i))&=\alpha(\sigma_1\sigma_2(i))\\ &=\alpha_*(\sigma_1)\ (\alpha(\sigma_2(i)))\\ &=\alpha_*(\sigma_1)\cdot\alpha_*(\sigma_2)\ (\alpha(i)). \end{aligned}$$

For $\sigma\in S_n$ and $\alpha\in C^n_m$, we put

$$\alpha(1)=i_1<\alpha(2)=i_2<\cdots<\alpha(m)=i(m)\leqq n$$

then $\{\sigma(i_1),\cdots,\sigma(i_m)\}\subset\{1,2,\cdots,n\}=N$. We rearrange $\{\sigma(i_1),\cdots,\sigma(i_m)\}$ by the order of natural number such that $\{1\leqq j_1<j_2<\cdots<j_m\leqq n\}$. We define $\sigma_*(\alpha)\in C^n_m$ by

$$\sigma_*(\alpha)\ (1)=j_1,\cdots,\sigma_*(\alpha)\ (m)=j_m.$$

Then there is a unique map $\alpha^*(\sigma)\in S_m$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\ \alpha\ } & N\\ {\scriptstyle\alpha^*(\sigma)}\downarrow & \copyright & \downarrow{\scriptstyle\sigma}\\ M & \xrightarrow{\ \sigma_*(\alpha)\ } & N \end{array}\ .$$

That is, if $\sigma(i)=j_k$ then $\alpha^*(\sigma)(1)=k$, $\sigma_*(\alpha)(k)=j_k$ and so on. Thus we can define a reduction map $\alpha^*:\ S_n\longrightarrow S_m$ for each $\alpha\in C^n_m$.

In general, α^* is not a group homomorphism as shown in the following example.

Example 2.1. With the above notations, for $\alpha\in C^n_m$, $\sigma,\tau\in S_n$ we have

$$\alpha^*(\sigma\cdot\tau)=(\tau_*(\alpha))^*\ (\sigma)\cdot\alpha^*(\tau)\in S_*.$$

Proof. To begin with, we note that

$$(\sigma\cdot\tau)_*\ (\alpha)=\sigma_*(\tau_*(\alpha)).$$

Suppose the commutative diagrams

$$\begin{array}{ccc} M & \xrightarrow{\ \alpha\ } & N \\ {\scriptstyle \alpha^*(\tau)}\downarrow & © & \downarrow{\scriptstyle \tau} \\ M & \xrightarrow{\ \tau_*(\alpha)\ } & N \\ {\scriptstyle \tau_*(\alpha)^*(\sigma)}\downarrow & © & \downarrow{\scriptstyle \sigma} \\ M & \xrightarrow{\ \sigma_*(\tau_*(\alpha))\ } & N \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{\ \alpha\ } & N \\ {\scriptstyle \alpha^*(\sigma\cdot\tau)}\downarrow & © & \downarrow{\scriptstyle \sigma\cdot\tau} \\ M & \xrightarrow{\ (\sigma\cdot\tau)_*(\alpha)\ } & N \end{array} .$$

Then, in view of $(\sigma\cdot\tau)_*(\alpha)=\sigma_*(\tau_*(\alpha))$ it follows that

$$(\sigma\cdot\tau)_*\ (\alpha)\cdot\alpha^*(\sigma\cdot\tau)=(\sigma\cdot\tau)\cdot\alpha=(\sigma\cdot\tau)_*\ (\alpha)\ (\tau_*(\alpha)^*(\sigma)\cdot\alpha^*(\tau)),$$

and hence $\alpha^*(\sigma\cdot\tau)=(\tau_*(\alpha))^*\ (\sigma)\cdot\alpha^*(\tau)$ as desired. ///

Proposition 2.2. For $\alpha\in C_*^*$, $\sigma\in S_n$ and $\nu\in S_*$

$$\alpha^*(\sigma\cdot\alpha_*(\nu))=\alpha^*(\sigma)\cdot\nu.$$

Proof. By example 2.1 we have

$$\alpha^*(\sigma\cdot\alpha_*(\nu))=(\alpha_*(\nu)_*(\alpha))^*(\sigma)\cdot\alpha^*(\alpha_*(\nu)).$$

Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \alpha\ } & N \\ {\scriptstyle \alpha^*(\alpha_*(\nu))}\downarrow & & \downarrow{\scriptstyle \alpha_*(\nu)} \\ M & \xrightarrow{\ \alpha_*(\nu)_*(\alpha)\ } & N \end{array} .$$

Since $\alpha_*(\nu)\ (\alpha(i))=\alpha(\nu(i))$, it is clear that $\alpha_*(\nu)_*(\alpha)=\alpha$ and $\alpha^*(\alpha_*(\nu))=\nu$. Thus we have

$$(\alpha_*(\nu)_*(\alpha))^*(\sigma)\cdot\alpha^*(\alpha_*(\nu))=\alpha^*(\sigma)\cdot\nu.\quad ///$$

Proposition 2.3. For each $\alpha \in C_m^n$, α^*: $S_n \longrightarrow S_m$ is a right S_m-map.

Proof. For each $\nu \in S_m$ and $\sigma \in S_n$, $\sigma \cdot \nu$ is defined to be

$$\sigma \cdot \alpha_*(\nu).$$

Then $\alpha^*(\sigma \cdot \nu) = \alpha^*(\sigma \cdot \alpha_*(\nu)) = \alpha^*(\sigma) \cdot \nu$ by the above proposition 2.2, which means that α^* is a right S_m-map. ///

Let $N-1$ be the set $\{1, 2, \cdots, n-1\}$. We define the inclusion map ρ: $N-1 \longrightarrow N$(for $i=1, \cdots, n-1$, $\rho(i)=i$). Then for each $\sigma \in S_{n-1}$ $\rho_*(\sigma)=\sigma$ is defined by

$$\text{for all } i=1, \cdots, n-1, \ \rho_*(\sigma(i))=i \text{ and } \rho_*(\sigma(n))=n.$$

Hence we may suppose that $S_{n-1} \subset S_n$. We shall use the notation $R=\rho^*$: $S_n \longrightarrow S_{n-1}$. Then R is characterized by

(1) it is a right S_{n-1}-map(proposition 2.3)

(2) $R(\tau_{k,n})=1$, for $\tau_{k,n}(1 \leqq k \leqq n)=(k, k+1, \cdots, n-1, n)$.

Definition 2.4. For a set X the action of S_n on the right $X^n=\{(x_1, \cdots, x_n) \mid i=1, \cdots, n$ $x_i \in X\}$ is defined by

$$(x_1, x_2, \cdots, x_n) \cdot \sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)}),$$

where $(x_1, \cdots, x_n) \in X^n$ and $\sigma \in S_n$. Similarly, for each $\alpha \in C_m^n$ the map α^*: $X^n \longrightarrow X^m$ is defined by

$$\alpha^*(x_1, \cdots, x_n) = (x_{\alpha(1)}, \cdots, x_{\alpha(m)}) \in X^m.$$

Suppose further that X has a base point $*$. Then α is said to be entire for $(x_1, \cdots, x_n)$ if $x_i = *$ for all $i \in N-\alpha(M)$.

Proposition 2.5. For $\alpha \in C_m^n$, $\sigma \in S_n$ and $x=(x_1, \cdots, x_n) \in X^n$

$$\alpha^*(x) \cdot \sigma \,|\, \alpha(M) = (\sigma_*(\alpha))^* \ ((x) \cdot \alpha^*(\sigma))$$

and α is entire for x if and only if $\sigma_*(\alpha)$ is entire for $x \cdot \alpha^*(\sigma)$.

Proof. We note that $\alpha^*(\sigma)(m+i)=m+i$, $i=1, 2, \cdots, r$, $m+r=n$. Therefore, $\alpha^*(\sigma) \in S_m$. In view of the commutative diagram below

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ {\scriptstyle \alpha^*(\sigma)} \downarrow & © & \downarrow {\scriptstyle \sigma} \\ M & \xrightarrow{\sigma_*(\alpha)} & N \end{array} .$$

We see that $\sigma_*(\alpha)^*(x\cdot\alpha(\sigma))=(x_{\alpha^*(\sigma)(1)},\cdots,x_{\alpha^*(\sigma)(m)},\ x_{m+1},\cdots,x_n)$

$$=(x_{\sigma_*(\alpha)(\alpha^*(\sigma)(1))},\cdots,x_{\sigma_*(\alpha)(\alpha^*(\sigma)(m))})$$
$$=(x_{\sigma\alpha(1)},\cdots,x_{\sigma(\alpha(m))})$$
$$=\alpha^*(x_1,\cdots,x_n)\cdot\sigma|\alpha(M).$$

The second part of our assertion is clear from the above expressions. ///

A simplicial set K is a graded set indexed on the non-negative integers together with maps

$$\partial i:\ K_q\longrightarrow K_{q-1},\quad S_i:\ K_q\longrightarrow K_{q+1},$$

$0\leqq i\leqq q$, which satisfy the following identities

$$\text{(i)}\ \partial_i\partial_j=\partial_{j-1}\partial_i \text{ if } i<j$$
$$\text{(ii)}\ S_iS_j=S_{j+1}S_i \text{ if } i\leqq j$$
$$\text{(iii)}\ \partial_iS_j=S_{j-1}\partial_i \text{ if } i<j$$
$$\partial_jS_j=S_{j-1}\partial_i \text{ if } i<j$$
$$\partial_jS_j=1=\partial_{j+1}S_j$$
$$\partial_iS_j=S_j\partial_{i-1} \text{ if } i>j+1.$$

The elements of K_q are called q-simplices, ∂_i and S_i are called a face and a degeneracy operators respectively. For two simplicial sets K and L, f: $K\longrightarrow L$ is a simplicial map if ① ${}^{\forall}q\geq 0$, f_q: $K_q\longrightarrow L_q$ ②

$$\begin{array}{ccc} K_q & \xrightarrow{f_q} & L_q \\ \partial_i\downarrow & & \downarrow\partial_i \\ K_{q-1} & \xrightarrow{f_{q-1}} & L_{q-1} \end{array} \quad\text{and}\quad \begin{array}{ccc} K_q & \xrightarrow{f_q} & L_q \\ S_i\downarrow & & \downarrow S_i \\ K_{q+1} & \xrightarrow{f_{q+1}} & L_{q+1} \end{array}$$

are commutative diagrams ③ $f=\{f_q|q\geq 0\}$.

Let $G(\neq\phi)$ be a discrete group. A simplicial set of groups (or simplicial groups) WG is defined as follows.

$$(WG)_n=G^{n+1}=\{<g_0,\cdots,g_n>|g_i\in G \text{ for } i=0,\cdots,n\}$$
$$\partial i<g_0,\cdots,g_n>=<g_0,\cdots,g_{i-1},\ \hat{g}_i,\ g_{i+1},\cdots,g_n>\ (\hat{g}_i \text{ means that } g_i \text{ is omitted.})$$
$$S_i<g_0,\cdots,g_n>=<g_0,\cdots,g_{i-1},\ g_i,\ g_i,\ g_{i+1},\cdots,g_n>$$

for all $0\leqq i\leq n$. In this case, G^{n+1} is a group with the group operation:

$$<g_0,g_1,\cdots,g_n>\cdot<g_0',\ g_1',\cdots,g_n'>=<g_0g_0',\ g_1g_1',\cdots,g_ng_n'>.$$

Moreover, G acts freely on the right of WG such that

$$<g_0,\ g_1,\cdots,g_n>\ g=<g_0g,\cdots,g_ng>,$$

where $g \in G$ and $<g_0, \cdots, g_n> \in G^{n+1}$. Since each $(WG)_n = G^{n+1}$ has the identity as a base point we say that WG is a pointed simplicial set (group). We define a pointed map

$$\sigma\colon (WG)_n \longrightarrow (WG)_{n+1}$$
$$<g_0, \cdots, g_n> \longrightarrow <1, g_0, \cdots, g_n>$$

where 1 is the identity of G. With this in mind we can prove that the pointed simplicial set WG is contractible([3]). Furthermore, we have the following note([3]):

Note 2.6. For $<g_0, \cdots, g_n> \in (WG)_n$ and $g \in G$ we define

$$<g_0, \cdots, g_n> \sim <g_0 g, \cdots, g_n g>.$$

Then "$\sim$" is an equivalence relation. Put

$$WG/\sim = \overline{W}G$$

then $\overline{W}G$ is a classifying space for G, *i.e.*, $\overline{W}G$ is equal to an Eilenberg-MacLane space $K(G,1)$. Moreover $G \subset WG$.

Note 2.7. W is a functor from the category of pointed sets to the category of pointed simplicial sets. For pointed sets A_1 and A_2 we have

$$W(A_1 \times A_2) = W(A_1) \times W(A_2)$$

since $\quad W(A_1 \times A_2)_n = W(A_1)_n \times W(A_2)_n.$

Therefore, for pointed sets $A, A_1, \cdots, A_n$ and a base point preserving function $f\colon A_1 \times \cdots \times A_n \longrightarrow A$, there corresponds

$$W(f)\colon W(A_1 \times \cdots \times A_n) = W(A_1) \times \cdots \times W(A_n) \longrightarrow W(A).$$

In particular, for maps

$$\alpha_*\colon S_m \longrightarrow S_n,\ \alpha^*\colon S_n \longrightarrow S_m$$

mentioned before, there correspond

$$W(\alpha_*)\colon WS_m \longrightarrow WS_n,\ W(\alpha^*)\colon WS_n \longrightarrow WS_m.$$

We shall put $W(\alpha_*) = \alpha_*$ and $W(\alpha^*) = \alpha^*$.

Definition 2.8. Let X be a pointed simplicial set. Then the following relations generate an equivalence relation on the disjoint union

$$U(X)=\coprod_{n\geq 0} WS_n\times X^n$$

(1) $(w,x)\sim(w\cdot\sigma,\ x\cdot\sigma)$ $(w\in WS_n,\ x\in X^n,\ \sigma\in S_n)$

(2) $(w,x)\sim(\alpha^*(w),\ \alpha^*(x))$ $(w\in WS_n,\ x\in X^n,\ \alpha\in C^n_m\ (n\geq m\geq 0))$

where α is entire for x. We put

$$\Gamma^+(X)=U(X)/\sim.$$

We denote the equivalence class of $(w,x)\in WS_n\times X^n$ by $[w,x]\in\Gamma^+(X)$. Note that S_0 is the trivial group $\{1\}$, if we put $1=(\langle 1\rangle,\langle 1,1\rangle,\cdots)=WS_0$, then $(1,\phi)\in WS_0\times X^0$. Thus $[1,\phi]\in\Gamma^+X$ is the canonical base point of Γ^+X.

We define $\varphi: U(X)\times U(X)\longrightarrow U(X)$ by

$$((\langle\sigma_0\rangle,\ \langle\sigma_1,\sigma_2\rangle,\cdots)\times(x_1,\cdots,x_n),\ (\langle\eta_0\rangle,\langle\eta_1,\eta_2\rangle,\cdots)\times(x_1',\cdots,x_m'))$$
$$\longrightarrow(\langle\sigma_0\cdot\eta_0\rangle,\ \langle\sigma_1\cdot\eta_1,\ \sigma_2\cdot\eta_2\rangle\cdots)\times(x_1,\cdots,x_n,\ x_1',\cdots,x_n')\in WS_{n+m}\times X^{n+m}$$

where $(\langle\sigma_0\rangle,\ \langle\sigma_1,\sigma_2\rangle,\cdots)\times(x_1,\cdots,x_n)\in WS_n\times X^n$ and $(\langle\eta_0\rangle,\ \langle\eta_1,\eta_2\rangle,\cdots)\times(x_1',\cdots,x_m')$ $\in WS_m\times X^m$. In particular, $\sigma_i\eta_i\in S_{n+m}$ for $\sigma_i\in S_n$ and $\eta_i\in S_m$ is defined by

$$\sigma_i\eta_i(r)=\sigma_i(r) \text{ for } 1\leq r\leq n$$
$$\sigma_i\eta_i(n+r)=n+\eta_i(r) \text{ for } 1\leq r\leq m.$$

Then $U(X)$ is a non-commutative monoid with identity $(1,\phi)$.
Therefore Γ^+X is also a non-commutative monoid with identity $[1,\phi]$.

3. The Functors Γ^+ and Γ

It follows from Note 2.6 that the discrete space S_n occurs naturally as a subspace of WS_n and so as a subspace of $U(X)$.

We define the natural embedding

$$i_X:\ X\longrightarrow\Gamma^+X$$

for a pointed simplicial set X by $i_X(x)=[1,x]$ for all $x\in X$, where $1\in S_1\subset WS_1$. By a pair (Γ^+X,X) we mean X embedded in Γ^+X by i_X.

Definition 3.1. We put $K=\{1,\cdots,k\}$ and $KN=\{1,2,\cdots,kn\}$. For integers i such that $1\leq i\leq k$ we define λ_i: $N\longrightarrow KN$ by

$$\lambda_i(j)=(i-1)n+j \qquad (j\in N).$$

Since

$$KN=\{1,2,\cdots,n;\ n+1,\cdots,2n;\ 2n+1,\cdots,3n;\cdots;(k-1)n+1,\cdots,kn\}$$

λ_i maps N to the i-th block of n elements of KN. Therefore $\lambda_i\in C_n^{kn}$ and thus we have a homomorphism

$$(\lambda_i)_*:\ S_n \longrightarrow S_{kn}$$

(see § 2). In view of Note 2.7 there exists a homomorphism

$$W((\lambda_i)_*):\ WS_n \longrightarrow WS_{kn}$$

for $i=1,\cdots,k$. We put $W(\lambda_i)_*=(\lambda_i)_*$ (or$=\lambda_i$).

We also define a homomorphism μ: $S_k\longrightarrow S_{kn}$ by

$$\mu(\sigma)\ ((i-1)n+j)=(\sigma(i)-1)+j$$

for $\sigma\in S_k$, $1\leq i\leq k$ and $1\leq j\leq n$. By Note 2.7, we see that

$$W(\mu)=\mu:\ WS_k\longrightarrow WS_{kn}$$

is a homomorphism.

With the above notations we may define a map

$$h_X:\ \coprod_{k\geq 0}\coprod_{n\geq 0} WS_k\times(WS_n\times X^n)^k\longrightarrow U(X)$$

as follows. For $w\in WS_k$, $\alpha_i=(w_i,\ x^i)\in WS_n\times X^n(x^i=(x_1{}^i,\cdots,x_n{}^i)\in X^n)$, $(w,\ \alpha_1,\cdots,\alpha_k)\in WS_k\times(WS_n\times X^n)^k$. Define

$$h_X(w,\alpha_1,\cdots,\alpha_k)=(\mu(w)\cdot\lambda_1(w_1)\cdot\cdots\cdot\lambda_k\ (w_k),\ x^1,\cdots,x^k),$$

where we identify $(X^n)^k$ with X^{kn} in the obvious way. Note that for

$$(\langle\sigma_0{}^1\rangle,\langle\sigma_1{}^1,\sigma_2{}^1\rangle,\cdots),\cdots,(\langle\sigma_0{}^{k+1}\rangle,\ \langle\sigma_1{}^{k+1},\sigma_2{}^{k+1}\rangle,\cdots)\in WS_{kn},\ (\langle\sigma_0{}^1\rangle,\ \langle\sigma_1{}^1,\sigma_2{}^1\rangle,\cdots)$$
$$\cdots(\langle\sigma_0{}^{k+1}\rangle\cdots)=(\langle\sigma_0{}^1\cdots\sigma_0{}^{k+1}\rangle,\ \langle\sigma_1{}^1\cdots\sigma_1{}^{k+1},\ \sigma_2{}^1\cdots\sigma_2{}^{k+1}\rangle,\cdots)\in WS_{kn}.$$

Therefore, h_X is well defined since $\mu(w),\cdots,\lambda_k(w_k)$ are in WS_{kn}.

Moreover h_X induces a natural map H_X: $\Gamma^+\Gamma^+X\longrightarrow\Gamma^+X$ ([3]).

That is, for a base point preserving function f: $X\longrightarrow Y$ the following diagram is commutative.

$$\begin{array}{ccc} \Gamma^+\Gamma^+(X) & \xrightarrow{H_X} & \Gamma^+(X) \\ {\scriptstyle \Gamma^+\Gamma^+(f)}\downarrow & © & \downarrow{\scriptstyle \Gamma^+(f)} \\ \Gamma^+\Gamma^+(Y) & \xrightarrow{H_Y} & \Gamma^+(Y) \end{array} \qquad (*)$$

where $\Gamma^+(f)\ [w, x_1, \cdots, x_n]=[w, f(x_1), \cdots, f(x_n)]$ for $w \in WS_n$ and $(x_1, \cdots, x_n) \in X^n$.

Proposition 3.2. Let X be a pointed space X. Then the following diagrams are commutative:

$$\begin{array}{ccccc} \Gamma^+(X) & \xrightarrow{\Gamma^+(i_X)} & \Gamma^+\Gamma^+(X) & \xleftarrow{i_{\Gamma^+(X)}} & \Gamma^+(X) \\ & {\scriptstyle 1_{\Gamma^+(X)}}\searrow & \downarrow{\scriptstyle H_X} & \swarrow{\scriptstyle 1_{\Gamma^+(X)}} & \\ & & \Gamma^+(X) & & \end{array} \qquad \begin{array}{ccc} \Gamma^+\Gamma^+\Gamma^+(X) & \xrightarrow{H_{\Gamma^+X}} & \Gamma^+\Gamma^+(X) \\ {\scriptstyle \Gamma^+(H_X)}\downarrow & & \downarrow{\scriptstyle H_X} \\ \Gamma^+\Gamma^+(X) & \xrightarrow{H_X} & \Gamma^+(X) \end{array} .$$

Proof. The commutativity of the second diagram is obvious by the functorial property of H_X: $\Gamma^+\Gamma^+(X) \longrightarrow \Gamma^+(X)$ (see the preceding diagram) (*). The commutativity of the first diagram can be proved as follows.

(1) The commutativity of the left triangle: For each $[w, x_1, \cdots, x_n] \in \Gamma^+X$, we have $\Gamma^+(i_X)=[w, ([1, x_1], \cdots, [1, x_n])]$ and thus

$$\begin{aligned} H_X[w, ([1, x_1], \cdots, [1, x_n])] &= [\mu(w)\lambda_1(1)\cdots\lambda_n(1), x_1, \cdots, x_n] \\ &= [w, x_1, \cdots, x_n]. \end{aligned}$$

Hence we have the commutative diagram

$$\begin{array}{ccc} \Gamma^+(X) & \xrightarrow{\Gamma^+(i_X)} & \Gamma^+\Gamma^+(X) \\ & {\scriptstyle 1_{\Gamma^+(X)}}\searrow \; © & \downarrow{\scriptstyle H_X} \\ & & \Gamma^+(X) \end{array} .$$

(2) The commutativity of the right triangle: For each $[w, x_1, \cdots, x_n] \in \Gamma^+(X)$,

$$\begin{aligned} H_X \circ i_{\Gamma^+(X)} &= H_X[1, [w, x_1, \cdots, x_n]] \\ &= [\mu(1) \cdot \lambda_1(w), x_1, \cdots, x_n] \\ &= [w, x_1, \cdots, x_n]. \end{aligned}$$

Therefore the following diagram is commutative:

$$\begin{array}{ccc} \Gamma^+\Gamma^+(X) & \xleftarrow{i_{\Gamma^+(X)}} & \Gamma^+(X) \\ H_X \downarrow & © & \swarrow 1_{\Gamma^+(X)} \\ \Gamma^+(X) & & \end{array} \quad . \quad ///$$

Let M be a monoid. By the universal group of M is meant a group UM which is universal with respect to homomorphisms from M to groups. That is, there exists a natural monoid homomorphism $U: M \longrightarrow UM$ such that for a group G if there is a monoid homomorphism f: $M \longrightarrow G$ then there exists a unique group homomorphism F: $UM \longrightarrow G$ such that $f = F \cdot U$.

By taking the universal groups of the monoids and monoid homomorphisms involved, we may define the universal simplicial group UM of a simplicial monoid M. Therefore given a pointed simplicial set X we define ΓX to be the universal simplicial group of the simplicial monoid Γ^+X. That is $U\Gamma^+X = \Gamma X$.

In consequence, we have the functors Γ^+ and Γ such that

$$\Gamma^+: \text{Pss} \longrightarrow SM$$

and

$$\Gamma: \text{Pss} \xrightarrow{\Gamma^+} SM \xrightarrow{U} SG,$$

where Pss = the category of pointed simplicial sets and simplicial maps which preserve base points

SM = the category of simplicial monoids and simplicial monoid homomorphisms

and SG = the category of simplicial groups and simplicial group homomorphism.

Lemma 3.3. Given a pointed space X, $\pi_0(\Gamma^+X) = Z^+\pi_0(X)$, the free abelian monoid on the pointed set $\pi_0(X)$, where $Z^+ = \{1, 2, 3, \cdots\}$.

Proof. Since $(WS_n)_0 = S_n$, the vertices $(\Gamma^+X)_0 = F^+X_0$, the free monoid on the set of vertices of X. Now given a simplicial set A, $\pi_0(A) = A_0/\sim$, where $\sim$ is the equivalence relation generated by

$$\partial_0 a \sim \partial_1 a \text{ for } a \in A_1.$$

Any 1-simplex $\xi \in \Gamma^+X$ may be written $\xi = [\langle 1, \sigma \rangle, x_1, \cdots, x_n]$, $\sigma \in S_n$, $x_i \in X_1$. Then $\partial_0 \xi = [\sigma, \partial_0 x_1, \cdots, \partial_0 x_n] = [1, \partial_0 x_{\sigma^{-1}(1)}, \cdots, \partial_0 x_{\sigma^{-1}(n)}]$, and $\partial_1 \xi = [1, \partial_1 x_1, \cdots, \partial_1 x_n]$. Thus we see that for $\pi_0(\Gamma^+X)$ the order of the coordinates does not matter and so the result is stated. ///

Theorem 3.4. For $X, Y \in Obj(\text{Pss})$ and $f, g\colon X \longrightarrow Y$ in Morph(Pss) with $f \simeq g$ (homotopic), we have $\Gamma^+(f) \simeq \Gamma^+(g)$ and $\Gamma(f) \simeq \Gamma(g)$.

Proof. Let SS be the category consisting of all simplicial sets and all simplicial maps. Then we have the functor

$$\begin{array}{ccc} \text{Pss} \times SS & \longrightarrow & \text{Pss} \\ \Cup & & \Cup \\ A \times X & & A \times | X = A \times X / * \times X \end{array}$$

where $X \neq \phi$. Then there is a natural transformation

$$\beta^+\colon \Gamma^+(A) \times B \longrightarrow \Gamma^+(A \times | B) \quad (A \in \text{Pss},\ B \in SS)$$

which is defined as follows.

Recall that $\Gamma^+(A)$ is an identification space (quotient space) of the monoid $U(A) = \coprod_{n \geq 0} WS_n \times A^n$. Let

$$\begin{array}{cccc} \Delta_n\colon & B & \longrightarrow & B^n \\ & \Cup & & \Cup \\ & b & \longmapsto & (b, \cdots, b) \end{array}$$

be the n-fold diagonal map. This induces a map

$$\begin{array}{ccc} WS_n \times A^n \times B & \longrightarrow & WS_n \times (A \times B)^n \\ & {\scriptstyle 1_{WS_n} \times \Delta_n} \searrow \quad \nearrow & \\ & WS_n \times A^n \times B^n & \end{array}$$

which makes following diagram commutative.

We note that the group S_n (see §2) acts on A^n, B^n and $(A \times B)^n$ after usual manner, that is,

$$(x_1, \cdots, x_n)\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)}),$$
$$((x_1 \times y_1), \cdots, (x_n \times y_n))\sigma = ((x_{\sigma(1)} \times y_{\sigma(1)}) \cdots, (x_{\sigma(n)} \times y_{\sigma(n)})),$$

for $(x_1, \cdots, x_n) \in A^n$ or B^n and

$(x_1 \times y_1), \cdots, (x_n \times y_n) \in (A \times B)^n$ and $\sigma \in S_n$.

We define β^+ by the following commutative diagram.

$$\begin{array}{ccc}
(\coprod_{n\geq 0} WS_n\times A^n)\times B=\coprod_{n\geq 0} WS_n\times(A^n\times B)\longrightarrow \coprod_{n\geq 0} WS_n\times(A\times B)^n\longrightarrow & \coprod_{n\geq 0} WS_n\times(A\times|B)^n \\
\downarrow & \downarrow \\
\Gamma^+(A)\times B \xrightarrow{\qquad\qquad\beta^+\qquad\qquad} & \Gamma^+(A\times|B).
\end{array}$$

Consider the equivalence relation in definition 2.8. One of the identifications gives $[w, x]=[w\sigma, x\sigma]$ in $\Gamma^+(A)$, where $w\in WS_n$, $x=(x_1, \cdots, x_n)\in A^n$ and $\sigma\in S_n$. By definition of β^+, we have

$$\begin{aligned}
&\beta^+([w, x]\times b)=[w, (x_1, b), \cdots, (x_n, b)] \\
&\beta^+([w\sigma, x\sigma]\times b)=[w\sigma, (x_{\sigma(1)}, b), \cdots, (x_{\sigma(n)}, b)=[w\sigma, ((x_1, b), \cdots, (x_n, b))\sigma] \\
&\qquad =[w, (x_1, b), \cdots, (x_n, b)]
\end{aligned}$$

and thus $\beta^+([w, x], b)=\beta^+([w\sigma, x\sigma], b)$ where $b\in B$. The other relations in definition 2.8 come from identifying $WS_{n+1}\times A^n\times *\longrightarrow WS_n\times A^n$ which is defined as follows.

Since WS_n is a subgroup of WS_{n+1}, we put $T'=WS_{n+1}/WS_n$.

Then $WS_{n+1}=WS_n\times T'$. Define

$$\begin{array}{rccc}
T: & WS_{n+1}=WS_n\times T' & \longrightarrow & WS_n \\
& \Cup & & \Cup \\
& w\times t' & \longmapsto & w.
\end{array}$$

Next, the projection

$$\begin{array}{rccc}
p_n: & A^n\times\{*\} & \longrightarrow & A^n \\
& \Cup & & \Cup \\
& (x_1, \cdots, x_n)\times\{*\} & \longmapsto & (x_1, \cdots, x_n).
\end{array}$$

Then $T\times P_n$: $WS_{n+1}\times A^n\times\{*\}\longrightarrow WS_n\times A^n$ is well defined.
Since $A^n\times\{*\}\times B\longrightarrow A^n\times\{*\}\times B^{n+1}=A^n\times B^n\times\{*\}\times B\longrightarrow(A\times|B)^n\times(\{*\}\times B)\subset (A\times|B)^{n+1}$, we have the commutative diagram:

$$\begin{array}{ccc}
WS_{n+1}\times A^n\times\{*\}\times B\longrightarrow & WS_{n+1}\times(A\times|B)^n\times(\{*\}\times B)\subset WS_{n+1}\times(A\times|B)^{n+1} \\
\downarrow T\times p_n\times 1_B \quad © & \downarrow T\times p_n \\
WS_n\times A^n\times B \longrightarrow & WS_n\times(A\times|B)^n.
\end{array}$$

Thus, it follows that β^+ is well-defined. And the naturality of β^+ is obvious from the construction. We have a map

$$\begin{array}{rccc}
\beta: & \Gamma(A)\times B & \longrightarrow & \Gamma(A\times|B) \\
& \pi_i(r_i)^{\eta_i}\times b & \longmapsto & \pi_i(\beta^+(r_i, b)^{\eta_i}),
\end{array}$$

which is an extension of β^+.

We put $f_0=f$ and $f_1=g$. Since $f_0\simeq f_1: X\longrightarrow Y$ for $I=[0,1]$ we have the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X\times I & \longrightarrow & \{*\}\times I \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow \\ Y & = & Y & \longleftarrow & * \end{array}$$

where F: $f_0\simeq f_1$ is a homotopy.

From the homotopy F we have the induced map F^b: $X\times|I\longrightarrow Y$ where $F^b(x\times|0)=f_0(x)$ and $F^b(x\times|1)=f_1(x)$. Therefore

$$\Gamma^+(F^b)\circ\beta^+:\ \Gamma^+(X)\times I\longrightarrow\Gamma^+(Y)$$

and

$$\Gamma(F^b)\circ\beta:\ \Gamma(X)\times I\longrightarrow\Gamma(Y)$$

are homotopies of $\Gamma^+(f_0)\simeq\Gamma^+(f_1)$ and $\Gamma(f_0)\simeq\Gamma(f_1)$ respectively. ///

4. Infinite Loop Spaces

Let the topological space be a compactly generated based countable CW-space in this chapter. If there are topological spaces $X_0, X_1, \cdots$ such that $X=X_0$ and X_i has the weak homotopy type of ΩX_{i+1} for all $i=0,1,2,\cdots$ then X is called an infinite loop space, where Ω is the loop functor. In this sense,

$$Q(X)=\varinjlim\ \Omega^n S^n X$$

defined in §2 is an infinite loop space, which is proved as follows.

Proof. To begin with, we note that for a sequence of Hausdorff space

$$\{*\}\subset Y_0\subset Y_1\subset\cdots\subset Y_n\subset\cdots$$

① the space $\varinjlim Y_n$ has the weak topology with respect to $\{Y_n\}$,

② for a compact space $I=[0,1]$,

$$(\varinjlim Y_n)^I=\varinjlim Y_n^I,\ \ f(0)=*=f(1)$$

where the space $Y_n{}^I = \{f : I \longrightarrow Y^n \mid f$ is continuous$\}$ has the compact open topology ([10]).

It suffices to construct spaces $E_m (m \geq 0)$ such that

$$E_0 = Q(X),\quad E_i \underset{\sim}{w} \Omega E_{i+1}\ (i \geq 0)$$

where $\underset{\sim}{w}$ means a weak homotopy equivalence. Put $E_m = Q(S^m X)$ for $m \geq 0$. Then

$$\begin{aligned}\Omega E_m &= \Omega Q(S^m X) = \Omega \varinjlim \Omega^n S^{m+n}(X) = (\varinjlim \Omega^n S^{m+n}(X))^I \\ &= \varinjlim (\Omega^n S^{m+n}(X))^I \text{ (by the above description)} \\ &= \varinjlim \Omega^{n+1} S^{n+1}(S^{m-1}X) = Q(S^{m-1}X) = E_{m-1}\ (m \geq 1).\end{aligned}$$

Hence $Q(X)$ is an infinite loop space. ///

We shall give some of examples of infinite loop spaces as follows.

Example 4.1. (i) Suppose the Eilenberg-MacLane space $K(\pi, n)$. Since $\Omega K(\pi, n+1) \underset{\sim}{w} K(\pi, n)$ $(n \geq 0)$ ([12]), it follows that $K(\pi, n)$ is an infinite loop space for any $n \in \{0, 1, 2, \cdots\}$.

(ii) Let $O(k)$ = the orthogonal group in k-dimensions, $U(k)$ = the unitary group of k-dimensions, $Sp(k)$ = the symmetric group in k-dimensions.

$$O = \varinjlim O(k),\quad U = \varinjlim U(k),\quad Sp = \varinjlim Sp(k).$$

Let BO, BU and BSp be classifying spaces of the groups O, U and Sp respectively. Then we have the following ([10], [26])

$$\begin{aligned}&O \underset{\sim}{w} \Omega(Z \times BO),\quad Z \times BO \underset{\sim}{w} \Omega(U/O),\quad U/O \underset{\sim}{w} \Omega(Sp/U) \\ &Sp/U \underset{\sim}{w} \Omega(Sp),\quad Sp \underset{\sim}{w} \Omega(Z \times BSp),\quad Z \times BSp \underset{\sim}{w} \Omega(U/Sp) \\ &U/Sp \sim \Omega(O/U) \text{ and } O/U \underset{\sim}{w} \Omega(O),\end{aligned}$$

where Z = the set of integers.

If we put $E_n = 0$ $n \equiv 0 \bmod 8$

$$\begin{aligned}&E_n = Z \times BO \quad n \equiv 1 \bmod 8 \\ &E_n = U/O \quad n \equiv 2 \bmod 8 \\ &E_n = Sp/U \quad n \equiv 3 \bmod 8\end{aligned}$$

$E_n = Sp \quad n\equiv 4 \mod 8$
$E_n = Z\times BSp \quad n\equiv 5 \mod 8$
$E_n = U/Sp \quad n\equiv 6 \mod 8$
$E_n = O/U \quad n\equiv 7 \mod 8$

then O is an infinite loop space. Since

$$U \underset{\sim}{w} \Omega(Z\times BU),\quad Z\times BU \underset{\sim}{w} \Omega(U)$$

U is also an infinite loop space. It is shown that $Z\times BO$ is also an infinite loop space in ([6], [24]).

(ii) Suppose the infinite loop space $\{E_n\}$ such that $E_0=Q(X)$, and $E_n=Q(S^nX)$, where X is a pointed Hausdorff topological space. We put $F_n=\underset{\overrightarrow{i}}{\lim}\ \Omega^i E_{n+i}$. Then

$$\Omega F_{n+1} \underset{\sim}{w} F_n,\quad E_n \underset{\sim}{w} F_n \text{ for all } n\geq 0.$$

Proof. $\Omega F_{n+1}=\Omega \underset{\overrightarrow{i}}{\lim}\ \Omega^i E_{n+1+i}=\underset{\overrightarrow{i}}{\lim}\ \Omega^{i+1}E_{n+i+1}$(by the above description) $=F_n$. That is, $\Omega F_{n+1}=F_n$, and thus F_0 is an infinite loop space.

On the other hand, for $m\geq 0$ and $n\geq 0$,

$$\pi_m(F_n)=\pi_m(\underrightarrow{\lim}\Omega^i E_{n+i})=\underrightarrow{\lim}\ \pi_m(\Omega^i E_{n+i})$$
$$\cong\underrightarrow{\lim}\ \pi_{m+i}(E_{n+i})=\pi_m(E_n)\ \ ([10]).$$

Thus for all $n\geq 0$, $E_n \underset{\sim}{w} F_n$. ///

Let G_i and H_i be groups and f_i: $G_i \longrightarrow H_i$ a group homomorphism for $i\geq 0$. $\{f_i | i=0,1,2,\cdots\}$ is called a k-isomorphism if f_i is an isomorphism for $i<k$ and f_k is an epimorphism.

For topological space X, Y and a continuous map f: $X\longrightarrow Y$, if f_{i*}: $\pi_i(X,x)\longrightarrow \pi_i(Y,f(x))$ is a k-isomorphism for each $x\in X$ then f is said to be k-equivalence. Given a topological space X, there is a CW-complex K by CW-approximation such that f: $K\longrightarrow X$ is a weak homotopy equivalence. Then the pair (K,f) is called a resolution of X.

Lemma 4.2. Given an $(n-1)$ connected $(n\geq 1)$ space Y there exists a CW-complex K that has no cells of dimension $<n$ except for a single 0-cell and (K,f) is a resolution of Y.

Proof. Our proof is divided into two steps.

Step I. Given $f: X \longrightarrow Y$ such that $f|A$ is an inclusion (X, Y are topological spaces, $A \subset X$) there is a commutative diagram where i is an inclusion, $\pi j = 1$ and $j\pi \sim 1$ (rel $i(A)$), $\sim$ means a homotopy, Z is a mapping cylinder of f.

$$\begin{array}{ccc} & & Z \\ & \overset{i}{\nearrow} & \pi \downarrow \uparrow j \\ X & \underset{f}{\longrightarrow} & Y \end{array}$$

Step II. Assume first that $n \geq 1$. By induction, we will construct an m-dimensional K^m such that $K^m \supset K^{m-1} \supset \cdots \supset K^0 = *$ and m-equivalences $f_m: (K^m, *) \longrightarrow (Y, *)$ such that $f_m | K^{m-1} = f_{m-1}$. We begin the induction with $* = K^0 = K^{n-1}$ and $f_{n-1}(*) = *$. Suppose now that we have constructed (K^m, f_m). Let Z be the mapping cylinder of $f_m (A = *)$ as in step I. Then we have a commutative diagram

$$\begin{array}{ccccc} & & \overset{i^*}{\longrightarrow} \pi_i(Z, *) & \longrightarrow \pi_i(Z, K^m, *) \longrightarrow \cdots \text{exact} \\ \cdots\longrightarrow \pi_{i+1}(Z, K^m, *) \longrightarrow & \pi_i(K^m, *) & \cong \quad \downarrow \pi_* & \\ & & (f_m)_* \longrightarrow \pi_i(Y, *) & \end{array}$$

which shows $\pi_i(Z, K^m, *) = 0$ for $i \leq m$. Let $\{f_\alpha | \alpha \in \Lambda\}$ generate $\pi_{m+1}(Z, K^m, *)$, $f_\alpha: (B_\alpha^{m+1}, S_\alpha^m, *) \longrightarrow (Z, K^m, *)$, where B_α^{m+1} is an $(m+1)$ dimensional unit ball and S_α^m is its boundary. We construct K^{m+1} as follows:

$$K^{m+1} = K^m \cup \amalg B_\alpha^{m+1} / x \sim f_\alpha(x) \text{ for } x \in S_\alpha^m \subset B_\alpha^{m+1}.$$

K^{m+1} is a closure finite cell complex and we give it the weak topology. Hence K^{m+1} is a CW-complex and K^m is a subcomplex.

Define $F: (K^{m+1}, K^m) \longrightarrow (Z, K^m)$ extending by $F|B_\alpha^{m+1} = f_\alpha$: $(B_\alpha^{m+1}, S_\alpha^m, *) \longrightarrow (Z, K^m, *)$ for each $\alpha \in \Lambda$. Define $f_{m+1} = \pi F$ where $\pi: Z \longrightarrow Y$.

Then $f_{m+1} | K^m = \pi | K^m = f_m$. Consider now the mapping cylinder Z' of F with $A = K^m$ as in step I and the diagram

$$\begin{array}{ccc} & \pi_i(Z', K^m, *) \longrightarrow \pi_i(Z', K^{m+1}, *) \longrightarrow \text{exact} & \\ & \uparrow i_* \quad \cong \quad \searrow \pi_* & \\ \cdots\longrightarrow \pi_{i+1}(Z', K^{m+1}, *) \longrightarrow & \pi_i(K^{m+1}, K^m, *) \underset{F_*}{\longrightarrow} \pi_i(Z, K^m, *) & \end{array}$$

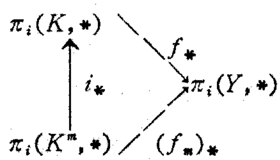
$$\pi_i(Z, K^m, *) = 0 \text{ for } i \leq m \text{ and } F_*: \pi_{m+1}(K^{m+1}, K^m, *) \longrightarrow \pi_{m+1}(Z, K^m, *)$$

is onto by construction.

Hence $\pi_i(Z', K^{m+1}, *)=0$ for $i\leq m+1$, and i_* is an $(m+1)$-isomorphism. Now consider π': $Z'\longrightarrow Z$ and π: $Z\longrightarrow Y$. These are homotopy equivalences and $\pi\pi'i=\pi F=f_{m+1}$. Hence $(f_{m+1})_*$ has the desired property and the induction is complete.

We now define $K=\coprod_{\alpha\in A} K^m$ and f: $K\longrightarrow Y$ by $f|K^m=f_m$. If we give K the weak topology, we have $\pi_i(K, K^m, *)=0$ for $i\leq m$.

Hence in the following diagram



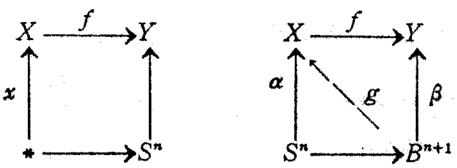
all maps are isomorphisms. ///

Lemma 4.3. Let X and Y be CW-complexes and assume that g: $X\longrightarrow Y$ is a weak homotopy equivalence. Then g is a homotopy equivalence.

Proof. As that of lemma 4.2, our proof is divided into two steps.

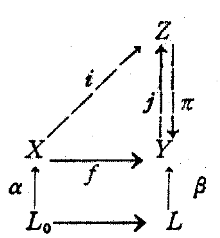
Step I. Let f: $X\longrightarrow Y$ be a base point preserving map. The map f is a weak homotopy equivalence iff given any CW-pair (L, L_0) and maps α: $L_0\longrightarrow X$, β: $L\longrightarrow Y$ with $f\alpha=\beta|L_0$, there is a map g: $L\longrightarrow X$ with $g|L_0=\alpha$ and $fg\sim\beta$(rel L_0).

For the proof of step I, consider the following diagrams, where B^{n+1} is the$(n+1)$-dimensional ball and S^n is n-sphere. Put $S^n=L_0$, $B^{n+1}=L$



leads to the conclusion that f_* is an isomorphism of homotopy groups by the null homotopic property for any choice of $x\in X$.

Suppose conversely that f induces isomorphisms in homotopy. Let Z be the mapping cylinder of f:



Define F: $L\times 0\cup L_0\times I\longrightarrow Z$ by $F(l,0)=j\beta(l)$ and $F(l,t)=(\alpha(l),t)$ for $l\in L_0$. Extend F to F: $L\times I\longrightarrow Z$. Let γ: $L\longrightarrow Z$ be a map given by $\gamma(l)=F(l,1)$. Then $\gamma(L_0)\subset X\times 1$. Produce g: $L\longrightarrow X\times 1$ with $g\sim\gamma(\text{rel } L_0)$. Now $g|L_0=\alpha$, and $fg=\pi g$ $\sim\pi\gamma\sim\beta(\text{rel } L_0)$ where the last homotopy is given by $(l,t)\longrightarrow\pi F(l,t)$.

Step II. Since g: $X\longrightarrow Y$ is a weak homotopy equivalence, by step I there exists h: $Y\longrightarrow X$ with $gh\sim 1$(homotopic) in the category C^* whose objects are topological space with a base point and morphisms are continuous functions which preserve the base point. Since (Y,h) is a resolution of X, we can similarly find j: $X\longrightarrow Y$ with $hj\sim 1$(homotopy) in C^*. Now $g\sim ghj\sim j$, hence $hg\sim 1$ and h is a homotopy inverse of g.

Theorem 4.4. Let X be an $(n-1)$connected based CW-complex of dimension$\leq$ $2n-1$, where $n\geq 1$, there is a CW-complex W such that $Q(X)$ has the same homotopy type as $Q(SW)$ and $\pi_n^s(X)=\pi_{n-1}^s(W)$.

Proof. By lemma 4.2, there exists a CW-complex Y with $(Y)^{n-1}$ a single point and f: $Y\longrightarrow X$ is a weak homotopy equivalence, i.e., $Y\sim X$ where "$\sim$" means a weak homotopy equivalence.

By lemma 4.3, $Y\approx X$ where "$\approx$" means the same homotopy type. Then construct W such that $SW\approx Y$ inductively by desuspending the attaching maps of the cells of Y. Since X and SW are CW-complexes with a base point, we see that $Q(X)\approx\Gamma(X)$, and $Q(SW)\approx\Gamma(SW)$ (see [3]). By theorem 3.4, $\Gamma(X)\approx\Gamma(SW)$, we have $Q(X)\approx Q(SW)$.

If $n>1$, $Y\approx SW$ is equivalent to $SY\approx S^2W$, thus the dimensional restriction implies that the adjoint map $Y\longrightarrow\Omega S^2Y$ induces an isomorphism in homology in dimension $\leq 2n-1$.

Finally,

$$\begin{aligned}\pi_n^s(X)=\pi_n(Q(X))\cong\pi_n(Q(SW))&=\pi_n(\varinjlim\Omega^n S^{n+1}W)\\ &=\pi_{n-1}(\Omega\varinjlim\Omega^n S^{n+1}W)\\ &=\pi_{n-1}(\varinjlim\Omega^{n+1}S^{n+1}W)\\ &=\pi_{n-1}(Q(W))=\pi_{n-1}^s(W)\end{aligned}$$

Definition 4.5. An$(n-1)$ connected space Y is called *atomic* if given any self map $f:Y\longrightarrow Y$ such that f_*: $H_n(Y)\longrightarrow H_n(Y)$ is an isomorphism then f_*: $H_*(Y)\longrightarrow H_*(Y)$ is an isomorphism. Y is called *H-atomic* if further, Y is an H-space and we are given only self H-maps f. Y is called ***atomic*** at p, where p is a prime, if Y has the same property with all homology groups having Z/pZ coefficients. If Y is called *H-atomic at p* if further, Y is an H-space and we are given only self-H-maps f.

Theorem 4.6. $Q(SX)$ is atomic at 2 where X has the same homotopy type as CP^n, $n\in\{1,2,\cdots\}$ with base point and $Q(SY)$ is atomic at 2 where Y has the same homotopy type as RP^n for $n\neq 3$ or 7 with base point.

Proof. $Q(CP^n)$ is H-atomic at 2 for $n\geq 1$, $Q(RP^n)$ is also atomic at 2 for $n\neq 3$ or 7 (see [8]). Since $\Omega Q(SX)$ is equivalent to $Q(X)$, thus $\Omega Q(SCP^n)$ is H-atomic at 2 for $n\geq 1$ and $\Omega Q(SRP^n)$ is also H-atomic at 2 for $n\neq 3$ or 7.

The "atomic property" is invariant under homotopy equivalence. If Y is 1-connected and ΩY is H-atomic (H-atomic at p) then Y is atomic (atomic at p).

By lemma 3.3 $\pi_0(\Gamma^+X)=Z^+\pi_0(X)$ which is the free abelian monoid on the pointed set $\pi_0(X)$. Since SCP^n and SRP^n are also path connected based CW-complexes, $Q(SCP^n)\approx\Gamma(SCP^n)$, $Q(SRP^n)\approx\Gamma(SRP^n)$.

Thus, it suffices to prove that $\pi_1(Q(SCP^n))=0$.

$$\begin{aligned}\pi_1(Q(SCP^n)=\pi_0(\Omega Q(SCP^n))&=\pi_0(\Omega\underrightarrow{\lim}\Omega^n S^n SCP^n)\\&=\pi_0(\underrightarrow{\lim}\Omega^{n+1}S^{n+1}CP^n)\\&=\pi_0(QCP^n)\\&=\pi_0(\Gamma^+(CP^n))\\&=Z^+\pi_0(CP^n)=0.\end{aligned}$$

Thus $Q(SCP^n)$ is atomic at 2. Furthermore, $Q(SCP^n)\approx\Gamma(SCP^n)\approx\Gamma(SX)\approx Q(SX)$ where $X\approx CP^n$, $n\in\{1,2,\cdots\}$.

Thus $Q(SX)$ is also atomic at 2. Similarly, $Q(SY)$ where $Y\approx RP^n$ $n\neq 3$ or 7 is also atomic at 2. ///

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