HONAM MATHEMATICAL JOURNAL
Volume 10, Number 1, August 1988

Remarks on D^n-Groups and Spectral Sequences*

by

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1. Introduction

In 1980 and 1983, it was proved that PD^2-groups are surface groups ([2],[3]). since then, topologists have been positively studing about PD^n-groups (or D^n-groups). For example, let a topological space X have a right π-action, where π is a multiplicative group. If each $x \in X$ has an open neighborhood U such that for each $U(=1) \in \pi$, $U_n U_x = \phi$, this right π_1-action is said to be proper. In this case, if X/π is compact then

(1) $\pi_1(X/\pi) \cong \pi(X$; connected, π_1: fundamental group) ([4]),

(2) if X is a differentiable manifold with demension n and ∂X(the boundary of X) $= \phi$ then

$$H^k(X;\ Z) \cong H_{n-k}(X;\ Z), \quad ([6])$$

where Z is the set of all integers. In paticular, since

$R^n(Z \times Z \cdots \times Z)$ (n-times) $= S \times S \times \cdots \times S$ (n-times) $= T^n$ is compact, $Z \times Z \times \cdots \times Z$ (n-times) is a PD^n-group over Z(Theorem 3.1)

Consider a short exact sequence of torsion free groups;

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1.$$

Then, for a left G-module A we have the spectral sequence

$$E_2^{q,n} \cong H^q(Q;\ H^n(N, A)) \xrightarrow[q]{} H^{n+q}(G, A) \quad (\text{see } \S 2).$$

Let K be the field of rationals. Our aim in the following; Under suitable Conditions.

(a) $E_2^{q,n+1} = E_2^{q+2,n}$

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* Research supported by Korean National Science and Engineering Foundation grant.
Received May 4, 1988.

(b) $0 \longrightarrow E_2^{q,n+r} \longrightarrow \cdots\cdots \longrightarrow E_2^{q+2r,n} \longrightarrow 0$ is exact (Theorem 3.3)

2. Preliminaries.

Let

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence of torsion free groups. Then, for a left G-module A we have a first quadrant spectral sequence $\{E_r, d_r\}$, natural in A with natural isomorphism.

$$E_2^{p,q} \cong H^p(Q, H(N, A)) \underset{p}{\longrightarrow} H^{p+q}(G, A) \qquad (*)$$

(Note that $E_2^{p,q} \underset{p}{\longrightarrow} H^{p+q}(G; A)$ means $E^{p,q}$, $H^{p+q}(G, A)$ ([1],[7]).

Let us denote the cohomological dimension of G by $\mathrm{cd}(G)$.

Proposition 2.1 In a short exact sequence of groups;

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

if $\mathrm{cd}(N)=n<\infty$ and $\mathrm{cd}(Q)=q<\infty$ then $\mathrm{cd}(G)<n+q$.

Proof. For a left G-module A we have the spectral sequence

$$E_2^{p,q} = H^p(Q, H^q(N, A)) \underset{p}{\longrightarrow} H^{p+q}(G, A)$$

as above(*)

$$H^i(N, A)=0 \text{ if } i>n \text{ and } H^j(Q,B)=0 \text{ if } j>q,$$

where A is a left G-module and B is a left Q-module. Thus

$$E_2^{r,s} = H^r(Q, H^s(N, A))=0 \text{ if } r>q \text{ or } s>n,$$

and hence

$$E_\infty^{r,s} = H^{r+s}(G, A)=0$$

This means that for every left G-module A.

$$H^{r+s}(G,A)=0 \text{ if } r+s>n+q$$

and thus $CD(G) \leq n+q \leq CD(N)+CD(Q)$. ⫽

Definition 2.2 A(multiplicative) group G is said to be of type (F, P) if there is a fite resolution over the trivial G-module Z by finitely generated projective modules, where Z is the set of integers.

Definition 2.3 A group G is a D^n-group over Z if and only if G satisfies the conditions

(1) G is of type (F,P)

(2) $H^k(G,Z[G]) = \begin{cases} 0, & \text{if } n \neq k \\ C, & \text{if } n=k, \end{cases}$

where $Z[G]$ is the group ring of G over Z.

(3) C is a Z-free module ([5],[8]).

A D^n-group sometimes is called a Poincare duality group of dimension n.

Throught this section, by a D^n-group we mean a D^n-group over Z without any statements.

Let G be a D^n-group. If $C=H^n(G,Z[G]) \cong Z$ then G is said to be an oriented Poincare duality group of dimension n, written PD^n.

Lemma 2.4. In the exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1,$$

if N is a D^n-group (PD^n-group) and Q is a D^q-group (PD^q-group) then G is a D^{n+q}-group (PD^{n+q}-group) such that

$$H^{n+q}(G, Z[G]) \cong H^q(Q, Z[Q]) \otimes H^n(N, Z[N]), \text{ where } \otimes = \otimes_Z.$$

Proof. By(*) above

$$E_2^{r,s} \cong H^r(Q, H^s(N, Z[G]) \underset{p}{\Longrightarrow} H^{r+s}(G, Z[G])$$

in our case. Since

$$H^n(N, Z[G]) \cong H^n(N, Z[N]) \otimes_{Z(N)} Z[G] \quad ([7], [9])$$

and

$$G/N = \{N, Nx_2, \cdots\} \quad (Q \cong \{1, x_2, x_3 \cdots\}$$

is a group (N is normal in G) we have

$$H^n(N, Z[G]) \cong H^n(N, Z[N] \otimes Z[Q]),$$

and thus

$$E_2^{r,s} \cong H^r(Q, H^s(N, Z[N]) \otimes Z[Q])$$

By Definition 2.3 and our assumption

$H^k(N, Z[N]) = 0$ if $k \neq n$ *and* $H^n(N, Z[N])$ is Z-free, and from

$$\begin{array}{ccccc} E^{r-2,n+1} & \longrightarrow & E^{r,n} & \longrightarrow & E^{r+2,n-1} \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

We have $E_2^{r,n} = E_3^{r,n} = \cdots\cdots = E_\infty^{r,n} \cong H^{r+n}(G, Z[G])$.

Since Q is a D^q-group (PD^q-group)

$$\begin{aligned} & H^r(Q, H^n(N, Z[N]) \otimes Z[Q]) \\ & \cong H_{q-r}(Q, H^q(Q, Z[Q])) \otimes_Z (Z[Q] \otimes H^n(N, Z[N])) \\ & \cong H_{q-r}(Q, H^q(Q, Z[Q]) \otimes H^n(N, Z[N])) \quad ([5], [9]), \end{aligned}$$

and thus

$$H^q(Q, H^n(N, Z[G])) \cong H_o(Q, H^q(Q, Z[Q]) \otimes H^n(N, Z[N]))$$

and

$$\begin{aligned} & H^q(Q, H^n(N, Z[G])) \cong H^{n+q}(G, Z[G]) \\ & H_o(Q, H^q(Q, Z[Q])) \otimes H^n(N, Z[N]) \cong H^q(Q, Z[Q] \otimes H^n(N, Z[N])). \end{aligned}$$

That is,

$$\begin{aligned} & H^k(G, Z)[G]) = 0 \text{ if } n-q \neq k \text{ and} \\ & H^{n+q}(G, Z[G]) \cong H^q(Q, Z[Q]) \otimes H^n(N, Z[N]). \end{aligned}$$

Note that $H^n(N, Z[N])$ and $H^q(Q, Z[Q])$ are free Z-modules.
Furthermore $H^{n+q}(G, Z[G])$ is Z-free.

Next, we have to prove that G is of type (F, P). Since N is a D^n-group(PD^n-group) and Q is a D^q-group(PD^q-group), We have projective resolutions;

$$0 \longrightarrow P_n(N) \xrightarrow{\delta'} \cdots \longrightarrow P_o(N) \longrightarrow Z \longrightarrow 0$$

$$0 \longrightarrow P_n(Q) \xrightarrow{\delta''} \cdots \longrightarrow P_o(Q) \longrightarrow Z \longrightarrow 0,$$

where $P_i(N)$ $(i=0,\cdots,n)$ and $P_j(Q)$ $(j=0,\cdots,q)$ are finitely generated.

Put

$$P_k(G) = \sum_{i+j=k} (P_i(N) \otimes P_j(Q)),$$

then $P_k(G)$ is finitely generated projective left $Z[G]$-module as follows. Since every element of G is represented by nx $(n \in N,\ x \in Q)$ by a unique way, if we define

$$nx(P_i(N) \otimes P_j(N)) = (nP_i(N) \otimes (xP_j(Q))$$

then $P_i(N) \otimes P_j(Q)$ is a left G-module. In fact,

$$Z[N] \otimes Z(Q) \xrightarrow{\cong} Z[G] \quad ((n \otimes x) \longmapsto nx)$$

is an isomorphism. For $n_1x_1,\ n_2x_2 \in G$, since

$$(n_1x_1)\ (n_2x_2) = (n_1n_2)\ (x_1x_2)$$

where $x_1n_2 = n_2x_1$, for $a \otimes b \in P_i(N) \otimes P_j(Q)$

$$(n_1x_1)\ (n_2x_2)\ (a \otimes b) = n_1n_2a \otimes x_1x_2b$$

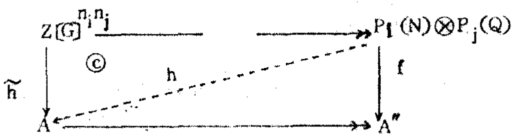
By our hypothesis there exist positive integers n_i, n_j, a $Z(N)$-epimorphism τ, and a $Z(Q)$-epimorphism σ such that

$$\tau:\ Z[N]^{n_i} \longrightarrow P_i(N)$$
$$\sigma:\ Z[Q]^{n_i} \longrightarrow P_j(Q).$$

Noting that $Z[N]$ and $Z[Q]$ are Z-free we have isomorphisms

$$Z[N]^{n_i} \otimes Z[Q]^{n_j} \simeq (Z[N] \otimes [Q])^{n_i n_j} \cong Z[G]^{n_i n_j}$$

For a $Z[G]$-epimorphism g: $A \longrightarrow A''$ and a $Z[G]$-homomorphism f: $P_i)N) \otimes P_j(Q) \longrightarrow A''$, suppose the following diagram:



Then there exists a $Z[G]$-homomorphism $\hat{h}$ such that $g \circ h = f \circ \tau \otimes \sigma$.

Define h: $p_i(N) \otimes P_j(Q) \longrightarrow A$ by $h \circ (\tau \otimes \sigma) = \hat{h}$. It follows that $g \circ h = f$. This means that each $P_i(N) \otimes P_j(Q)$ is a projective $Z[G]$-module. Therefore we have a projective semiexact sequence over Z as $Z[G]$-modules such that

$$P(G):\ 0 \longrightarrow P_n(N) \otimes P_q(Q) \xrightarrow{\delta} \cdots \longrightarrow \sum_{i+j=k} P_i(N) \otimes P_j(Q)$$
$$\longrightarrow \cdots \longrightarrow P_o(N) \otimes P_o(Q) \longrightarrow Z \otimes Z = Z \longrightarrow 0,$$

where $\delta = \delta' + \delta''$ and for $a \otimes b \in P_i(N) \otimes P_j(Q)$

$$\delta'(a \otimes b) = \delta' a \otimes b,\ \ \delta''(a \otimes b) = (-1)^i\ a \otimes \delta'' b.$$

For $a \otimes b \in P_i(N) \otimes P_j(Q)$ we assume that

$$\delta(a \otimes b) = \delta' a \otimes b + (-1)^i a \otimes \delta'' b = 0,$$

which is equivalent to $a \in \mathrm{Ker}\delta'$ and $b \in \mathrm{Ker}\delta''$. Thus there are $a' \in P_{i+1}(N)$, $b' \in p_{j+1}(Q)$ such that $\delta' a' = a$ and $\delta'' b' = b$.
Thus, $a' \otimes b \in P_{i+1}(N) \otimes P_j(Q)$ and

$$\delta(a' \otimes b) = \delta'\, a' \otimes b + (-1)^{i+1} a' \otimes \delta''\, b.$$

Since $\delta''(b) = \delta''\, \delta''(b') = 0$ and $\delta' a' = a$ we have $\delta\ (a' \otimes b) = a \otimes b$.

That is, the complex $P(G)$ is exact. Consequently G is of type (F, P). If N and Q are PD-groups then

$H^n(N, Z[N]) \cong Z \cong H^q(Q, Z[Q])$, and thus
$H^{n+q}(G, Z[G]) \cong Z \otimes Z = Z$. ///

3. Main Theorems

From Lemma 2.4, we obtain the following:

Theorem 3.1. $Z\times Z\times\cdots\times Z$ (n-times) is a PD^n-group over Z.

Proof. We denote each element $n\in Z$ by$[N]$ and define $a$multiplication by$[M][N]=[M+N]$. Therefore the group ring $Z[Z]$ is well defined. In $Z(Z)$

$$\sum n_i[a_i]+\sum m_i[a_i]=\sum(n_i+m_i)[a_i]$$

and

$$\sum n_i[a_i]\cdot\sum m_j[b_j]=\sum_{ij} n_i m_j[a_i+b_j].$$

In particular, Z is the trivial $Z[Z]$-module, *i.e.*, for $m\in Z$ and $n[a]\in Z[Z](n[a]m=nm$. Suppose a $Z[Z]$-module sequence.

$$(\sigma\text{-}K):\ 0\longrightarrow Z[Z]\xrightarrow{\sigma}Z[Z]\xrightarrow{k}Z\longrightarrow 0.$$

is defined as follows:

$$\sigma([m])=[m]-[0]$$
$$\sigma(m_1+m_2)=[m_2]([m_1]-[-m_2])$$
$$([m]\sigma)([n])=[m]([n]-[-m]).$$

Then it follows that

$$m=n_1+n_2\Longrightarrow\sigma[m]=\sigma[n_1+n_2]$$
$$([m]\sigma)[n]=\sigma([m][n])=\sigma([m+n]).$$

Thus σ is a $Z[Z]$-monomorphism. K is defined by $K(n[m])=n$ for $n[m]\in Z[Z]$ Then K is a $Z[Z]$-epimorphism. In particular, $\mathrm{Ker}K=\{\sum n_i[m_i][a_i]-[0]\}n_i[m_i],\ a_i\in Z[Z]$ aud thus Im $\sigma=\mathrm{Ker}K$. Therefore the sequence is exact as $Z[Z]$-modules.

Hence the multiplicative group Z is of type (F,P). From the sequence$(\sigma\text{-}K)$ we get the following.

$$\begin{array}{l}0\longleftarrow \mathrm{Hom}_{Z(Z)}(Z[Z],\ Z[Z])\xleftarrow{\sigma^*}\mathrm{Hom}\ (Z[G],\ Z[G])\xleftarrow{K^*}\\ \qquad\qquad\qquad\qquad \wr\| \qquad\qquad\qquad\qquad\qquad \wr\| \\ \qquad\qquad\qquad\qquad Z[G] \qquad\qquad\qquad\qquad\quad Z[G] \\ \mathrm{Hom}_{Z(Z)}(Z,Z[Z])\longleftarrow 0 \\ \qquad\qquad \| \\ \qquad\qquad 0\end{array}$$

Therefore,

$$H^k(Z,Z[Z])=\begin{cases}0, & \text{if } k\neq 1\\ Z, & \text{if } k=1\end{cases}$$

and it follows that the multiplicative group Z is a PD^1-group over Z.

In general, $Z\times Z\times\cdots\times Z$($n$-times) is *a* multiplicative group with multiplication

$$([m_1],\cdots,[m_u])\cdot([n_1],\cdots,[n_n])=m_1n_1,\cdots,$$
$$([m_n][n_n])=[(m_1+n_1),\cdots,[m_n+n_n]).$$

We can use mathematical induction to complete our proof. Assume $Z\times Z\times\cdots\times Z((n-1)$ -times) is a PD^{n-1}-group over Z. In exact sequence

$$1\longrightarrow Z\times\cdots\times Z\ \ ((n\text{-}1)\text{-times})\longrightarrow Z\times\cdots\times Z(n\text{-times})\longrightarrow Z\longrightarrow 1$$
$$[[m_1],\cdots,[m_{n-1}]]\longrightarrow([m_{n-1}],\cdots,[0])\longrightarrow[m_n]$$

Since $Z\times\cdots\times Z((n-1)$-times) is a PD^{n-1}-group over Z and Z is PD^1-group over Z by Lemma 2.4 $Z\times\cdots\times Z$(n-times) is a PD^n-group over Z. ///

Let us put K the set of all rationals. *A* D^n-group G over K is defined by

(1) G is of type (F,P) over K

(2) $H^i(G,K[G])$ is zero if $i\neq n$

(3) $H^n(G,K[G])$ is K-free, where $K[G]$ is the group ring of G over K.

Throughout this section we assume that every group is torsion free over K.

The following is well-known

Lemma 3.2. If a group G is a D^n-group over Z, then G is also a D^n-group over K.

Let

$$1\longrightarrow N\longrightarrow G\longrightarrow Q\longrightarrow 1 \qquad (**)$$

be a short exact sequence of groups. Then by (*) in §2 we have the spectral sequence

$$E_2^{r,s}\cong H^r(Q,H^s(N,K[G]))\underset{r}{\longrightarrow}H^{r+s}(G,K[G])$$

Since $K[G]$ is $K[N]$-free.

$$H^s(N,K[G])\cong H^s(N,K[N])\otimes_k K[Q],$$

and since every K-module is K-free $H^s(N,K[N])\otimes_k K[Q]$ is $K[Q]$-free. Thus we have the following ([6],[9]):

$$\begin{aligned}H^r(Q,H^s(N,K[G]))&\cong H^r(QH^s(N,K[N]\otimes_k K[Q])\\&\cong H^r(Q,K[Q])\otimes_{k(Q)}(H^s(N,K[N]\otimes_k K[Q])\\&\cong H^r(Q,K[Q])\otimes_k H^s(N,K[N])\\&\cong E_2^{r,s}\end{aligned}$$

In (**) we assume that

(1) N is of type (F, P) over Z

(2) G is a D^{n+q}-group over Z

(3) $cdZ(Q) < \infty$.

Then we can prove that Q is of type (F, P) over Z ([9]).

Theorem 3.3. Under the above situation let us put

$$m = \text{least number with } H^m(N, K[N]) \neq 0$$
$$P = \text{least number with } H^p(Q, K[Q]) \neq 0.$$

Then we have the following:

(1) $m = n$ and $p = q$

(2) There is an isomorphism $E_2^{q,n+1} \cong E_2^{p+2,n}$

(3) There is an exact sequence

$$0 \longrightarrow E_2^{q,n+r} \longrightarrow \cdots \longrightarrow E_2^{q+2r,n} \longrightarrow 0$$

Proof. (1): By the spectral sequence

$$E^{r,s} \cong H^r(Q, K[Q]) \otimes_k H^s(N, K[N])$$

and by our assumptions

$$E_2^{p,m}(\neq 0) \cong H^{p+m}(G, K[G]).$$

By Lemma 3.2, G is a D^{n+q}-group over K and thus $p = q$, $m = n$

(2): In our spectral sequence

$$\left.\begin{array}{l} E_2^{q-2,n+2} \longrightarrow E_2^{q,n+1} \xrightarrow{d_2} E_2^{q+2,n} \\ \| \\ 0 \\ E_3^{q-3,n+3} \longrightarrow E_3^{q,n+1} \longrightarrow E_3^{q+2,n-1} \\ \| \qquad\qquad\quad \| \qquad\qquad | \\ 0 \qquad\quad \ker d_2 \qquad\quad 0 \end{array}\right\} \Rightarrow \begin{array}{l} E_3^{q,n+1} = \cdots = E_\infty^{q,n+1} \\ \cong H^{q+n+1}(G, K[G]) = 0 \end{array}$$

Thus $E_3^{q,n+1} = \text{Ker } d_2 = 0$. From

$$\left.\begin{array}{l} E_2^{q,n+1} \longrightarrow E_2^{q+2,n} \longrightarrow E_2^{q+4,n-1} \\ \qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\qquad\quad 0 \\ E_3^{q-1,n+2} \longrightarrow E_3^{q+2,n} \longrightarrow E_3^{q+5,n-2} \\ \| \qquad\qquad\quad \| \qquad\qquad \| \\ 0 \qquad E_2^{d+2,n}/\text{Im } d_2 \qquad 0 \end{array}\right\} \Rightarrow E^{q+2,n}/\text{Im } d_2 = 0.$$

We have the isomorphism d_2: $E_2^{q,n+1} \cong E_2^{q+2,n}$

(3): Since

$$\begin{array}{ccccc} E_2^{q-1,n+1} & \longrightarrow & E_2^{q+1,n} & \longrightarrow & E_2^{q+3,n-1} \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

and $H^n(N, K[N]) \neq 0$, from

$$E^{q+1,n} \cong H^{q+1}(Q, K[Q]) \otimes_k H^n(N, K[N]) \cong E_\infty^{q+1,n}$$
$$\cong H^{q+1+n}(G, K[G]) = 0$$

$H^{q+1}(Q, K[Q]) = 0$. Similarly, from

$$\begin{array}{ccccc} E_2^{q+1,n} & \longrightarrow & E_2^{q+3,n} & \longrightarrow & E_2^{q+5,n-1} \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

$$H^{q+3}(Q, K[Q]) = 0.$$

In consequence, $H^{q+i}(Q, K[Q]) = 0$ for $i = 1, 3, 5, \cdots$

Therefore we have

$$E_3^{q+5,n+t} = 0$$

because that

(1) S: odd $\Longrightarrow E_2^{q+5,n+t} = 0 \Longrightarrow E_3^{q+5,n+t} = 0$

(3) S: even $\Longrightarrow$ $$\begin{array}{ccccc} E_3^{q+s-3,n+t+2} & \longrightarrow & E_3^{q+t,n+t} & \longrightarrow & E_3^{q+s+3,n+t-2} \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

$$\Longrightarrow E_3^{q+s,n+t} = E_3^{q+s,n+t} \cong H^{q+s+n+t}\ (G, K[G]) = 0$$

According, we get the exact sequence

$$0 \longrightarrow E_2^{q,n+r} \longrightarrow E_2^{q+2,n+r-1} \longrightarrow \cdots \longrightarrow E_2^{q+2r,n} \longrightarrow 0. \quad /\!/\!/$$

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