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The Cup and Cap Product of Simplicial Modules

by

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The concepts of simplicial sets arose in the study by Eilenberg and Zilber ([3], [4]), and simplicial abelian groups can be found in the paper [2] by Eilenberg and MacLane. In [1], Dold gives an equivalence between the categories of simplicial modules and chain complexes of modules. As in [6], the cup products for these simplicial modules are defined but even so the cap products can not define.

In this paper we shall give a definition of the cap products of new simplicial modnles (Definition 2) and prove a relation between cup and cap products of simplicial modules (Theorem 3).

Throughout this paper, by Λ and $\otimes$ we mean commutative rings with identity and $\otimes_\Lambda$, respectively. The objets $[p]=\{0,1,\ldots,p\}$ with morphisms all weakly monotonic maps μ ([6]) constitute a category M. Let mod_Λ be the category consisting of all Λ-modules and all Λ-module homomrphisms.

A simplicial Λ-module S is a contravariant functor from M to mod_Λ. If we put $S([p])=S_p$ (a Λ-module) then there are Λ-module homomorphisms

$$d_i^q:\ S_q \longrightarrow S_{q-1},\quad S_i^q:\ S_q \longrightarrow S_{q+1}$$

for all $i=0,1,\ldots,q$ ($q>0$ in the case of d_i^q), which satisfy the identities ([6]):

$$\left.\begin{array}{lll} d_i^{q-1}d_j^q = d_{j-1}^{q-1}d_i^q, & i<j & \\ S_i^{q+1}S_j^q = S_{j+1}^{q+1}S_i^q, & i\leq j & \\ d_i^{q+1}S_j^q = S_{j-1}^{q-1}d_i^q, & i<j & \\ \quad = 1_{S_q}, & i=j,\ i=j+1 & \\ \quad = S_j^{q-1}d_{i-1}^q, & i>j+1 & \end{array}\right\} \quad (*)$$

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Moreover,

$$\left.\begin{array}{ll} d_i^{q-1}d_j^q=d_j^{q-1}d_{i+1}^q, & i\geq j \\ S_i^{q+1}S_j^q=S_j^{q+1}S_{i-1}^q, & i>j \end{array}\right\} (*)'$$

Let $K(S)$ be the chain complex such that

$$\begin{array}{cccccc} K(S):\ \ldots\longrightarrow & K_q(S) & \xrightarrow{\partial_q} & K_{q-1}(S) & \longrightarrow\ldots\longrightarrow & K_0(S) \\ & \| & & \| & & \| \\ & S_q & & S_{q-1} & & S_0, \end{array}$$

where

$$\partial_q=d_0^q-d_1^q+\ldots+(-1)^q d_q^q:\ K_q(S)\longrightarrow K_{q-1}(S)$$

with $\partial_{q-1}\partial_q=0$ $(q\geq 2)$ (by $(*)$ and $(*)'$ above).

Let S: $M\longrightarrow \mathrm{mod}_\Lambda$ be a fixed simplicial Λ-module. Then we have a covariant functor U: $M\longrightarrow \mathrm{mod}_\Lambda$ such that for each $[n]\in M$

$$U([n])=\mathrm{Hom}_\Lambda\ (K_n(S),\Lambda)=U^n.$$

Let us put

$$e_i^q=\mathrm{Hom}_\Lambda(d_i^q, 1_\Lambda):\ U^{q-1}\longrightarrow U^q$$

and

$$t_i^q=\mathrm{Hom}_\Lambda(S_i^q, 1_\Lambda):\ U^{q+1}\longrightarrow U^q.$$

We have the chain complex;

$$U:\ U^0\longrightarrow U^1\longrightarrow\ldots\longrightarrow U^{q-1}\xrightarrow{\delta^q} U^q\longrightarrow\ldots,$$

where $\delta^q=e_0^q-e_1^q+\ldots+(-1)^q e_q^q$: $U^{q-1}\longrightarrow U^q$ and $\delta^q\delta^{q-1}=0$, because that for $\delta^q=\mathrm{Hom}_\Lambda(\partial_q, 1_\Lambda)$

$$\delta^q\delta^{q-1}=\mathrm{Hom}_\Lambda(\partial_q, 1_\Lambda)\cdot\mathrm{Hom}_\Lambda(\partial_{q-1}, 1_\Lambda)=\mathrm{Hom}_\Lambda(\partial_{q-1}\partial_q, 1_\Lambda)=0.$$

For each Λ-module A we define the new simplicial Λ-module $\tilde{A}: M \longrightarrow \mathrm{mod}_\Lambda$ such that for each $q \in Obj(M)$

$$\begin{aligned}\tilde{A}([q]) &= \tilde{A}_q \\ &= \text{the free } \Lambda\text{-module generated by the set } \mathrm{Hom}_\Lambda(U^q, A).\end{aligned}$$

If we put

$$d_i^q = \mathrm{Hom}_\Lambda(e_i^q, 1_A): \ \tilde{A}_q \longrightarrow \tilde{A}_{q-1}$$

and

$$S_i^q = \mathrm{Hom}_\Lambda(t_i^q, 1_A): \ \tilde{A}_q \longrightarrow \tilde{A}_{q+1},$$

then d_i^q and S_i^q satisfy the identities in (*) and (*)′. There is a chain complex

$$\begin{array}{ccccccc} K(\tilde{A}): \ \ldots \longrightarrow & K_q(\tilde{A}) & \xrightarrow{\partial_q} & K_{q-1}(\tilde{A}) & \longrightarrow \ldots \longrightarrow & K_0(\tilde{A}) \\ & \| & & \| & & \| \\ & \tilde{A}_q & & \tilde{A}_{q-1} & & \tilde{A}_0, \end{array}$$

where $\partial_q = d^q{}_0 - d^q{}_1 + \ldots + (-1)^q d_q^q: \ \tilde{A}_q \longrightarrow \tilde{A}_{q-1}$ with $\partial_{q-1}\partial_q = 0$ for $q \geq 2$.

For another Λ-module B, consider the chain complex $K(\tilde{B})$ which is defined as above. Then the Eilenberg-Zilber theorem says that there is a natural equivalence

$$K(\tilde{A} \times \tilde{B}) \underset{K}{\overset{\rho}{\rightleftarrows}} K(\tilde{A}) \otimes K(\tilde{B}) \quad ([5],[6]).$$

In particular, for each $a \in K_n(\tilde{A}) = \tilde{A}_n$ and $b \in K_n(\tilde{B}) = \tilde{B}_n$ the Alexander-Whiteney maps are defined as follows:

$$\left.\begin{aligned}\rho(a \times b) &= \sum_{i=0}^{n} d_{i+1}^{i+1} \ldots d_n^n a \otimes d_0^{n-i+1} \ldots d_{i-1}^n b \\ &= \sum_{i=0}^{n} a e_n^n \ldots e_{i+1}^{i+1} \otimes b e_{i-1}^n \ldots e_0^{n-i+1}\end{aligned}\right\} \ (**)$$

and for $a \in \tilde{A}_p$, $b \in \tilde{B}_q$ $(p+q=n)$

$$\left.\begin{aligned}K(a \otimes b) &= \sum_{(\mu,\nu)} (-1)^{\varepsilon(\mu)} (S_{\nu_q}^{n-1} \ldots S_{\nu_1}^p a \times S_{\mu_p}^{n-1} \ldots S_{\mu_1}^q b) \\ &= \sum_{(\mu,\nu)} (-1)^{\varepsilon(\mu)} [a t_{\nu_1}^p \ldots t_{\nu_q}^{n-1} \times b t_{\mu_1}^q \ldots t_{\mu_p}^{n-1}]\end{aligned}\right\} \ (**)'$$

where (μ,v) is a (p,q)-shuffle and $\varepsilon(\mu)=\sum_{i=1}^{p}\mu_i-(i-1)$ ([6]).

Furthermore, for each $f\in\mathrm{Hom}_\Lambda(U^0,A)$ if we define a Λ-module homomorphism

$$\begin{array}{ccc} \varepsilon\colon K_0(\tilde{A}) & \longrightarrow & \Lambda \\ \Cup & & \Cup \\ f & \longmapsto & 1, \end{array}$$

then it follows that $K(\tilde{A})$ is an augmented chain complex for each Λ-module A.

We define homology and cohomology groups:

$$\mathrm{H}_n(K(\tilde{A}))=H_n^S(A)$$

$$H^n(\mathrm{Hom}_\Lambda(K(\tilde{A}),\Lambda)=H_S^n(A)$$

for all $n=0,1,2,\cdots$ For each $n\geq 0$ we put

$$\tilde{U}_n=\mathrm{Hom}_\Lambda(\tilde{U}^n,A)$$

and

$$K_n(\tilde{U})=\text{the free } \Lambda\text{-module generated by } \tilde{U}_n.$$

As before, there is a chain complex such that

$$\begin{array}{llll} K(\tilde{U})\colon \ldots \longrightarrow & K_q(\tilde{U}) \xrightarrow{\partial_q} K_{q-1}(\tilde{U}) & \longrightarrow \ldots \longrightarrow & K_0(\tilde{U}) \\ & \parallel & & \parallel \\ & K_q(\tilde{A}) \xrightarrow{\partial_q} K_{q-1}(\tilde{A}) & & K_0(\tilde{A}). \end{array}$$

The simplicial diagonal map $\tilde{U}\longrightarrow\tilde{U}\times\tilde{U}\,(f\longmapsto f\times f)$ induces the chain transformation

$$\triangle\colon K(\tilde{U})\longrightarrow K(\tilde{U}\times\tilde{U}).$$

Moreover, there exists the chain transformation

$$\begin{array}{lccc} \omega=\rho\triangle\colon & K(\tilde{U}) \xrightarrow{\triangle} & K(\tilde{U}\times\tilde{U}) \xrightarrow{\rho} & K(\tilde{U})\otimes K(\tilde{U}) \\ & \parallel & & \parallel \\ & K(\tilde{A}) & & K(\tilde{A})\otimes K(\tilde{A}). \end{array}$$

Let us put

$$H^n(\mathrm{Hom}_\Lambda(K(\tilde{A})\otimes K(\tilde{A}),\Lambda)) = H^n_S(K(\tilde{A})\otimes K(\tilde{A}),\Lambda).$$

The *cross product*

$$P^H\colon H^k_S(A)\otimes H^m_S(A)\longrightarrow H^{k+m}(K(\tilde{A})\otimes K(\tilde{A}),\Lambda)$$

is defined by $[u]\otimes[v]\longmapsto[u\otimes v]$ for each

$$[u]\otimes[v]\in H^k_S(A)\otimes H^m_S A,$$

where $[u]$ is the cohomology class containing u.

Definition 1. With the above notations the composite $U=\omega^* P^H$, *i.e.*,

$$\begin{array}{ccc} H^k_S(A)\otimes H^m_S(A) & \xrightarrow{\quad U \quad} & H^{k+m}_S(A) \\ & {\scriptstyle P^H}\searrow \quad © \quad \nearrow{\scriptstyle \omega^*} & \\ & H^{k+m}_S(K(\tilde{A})\otimes K(\tilde{A}),\Lambda) & \end{array}$$

is called the *initial simplicial cup product.*

In fact, for $[u]\in H^k_S(A)$, $[v]\in H^m_S(A)$ and $\sigma\in K_{k+m}(\tilde{A})$, $u\cup v\in\mathrm{Hom}_\Lambda(K_{k+m}(\tilde{A}),A)$

and $$(u\cup v)(\sigma)=u(\sigma e^{k+m}_{k+m}\dots e^{k+1}_{k+1})\cdot v(\sigma e^{k+m}_{k-1}\dots e^{m+1}_0) \qquad (***)$$

by (**) above.

Moreover, $[u]\cup[v]=[u\cup v]$, $([u]\cup[v])(\sigma)=(u\cup v)(\sigma)$.

As in the case of topological spaces our cup products satisfy the following properties:

1°. By the naturality of cup products, for each Λ-module homomorphism $f\colon B\longrightarrow A$ and $[u]\cup[v]\in H^{k+m}_S(A)$

$$f^*([u]\cup[v])=f^*([u])\cup f^*([v]).$$

2°. For $1\in H^0_S(A)$ and $[u]\in H^n_S(A)$

$$1\cup[u]=[u]\cup 1=[u].$$

Proof. We first note that 1 is the cohomology class containing the augmentation $\varepsilon: K_0(\tilde{A}) \longrightarrow A$.

Take an element $\sigma \in K_n(\tilde{A})$, then by (***),

$$\begin{aligned}([u] \cup 1)(\sigma) = (u \cup \varepsilon)(\sigma) &= u(\sigma) \cdot \varepsilon(\sigma e^n_{n-1} \dots e^1_0) \\ &= u(\sigma) \\ &= [u](\sigma),\end{aligned}$$

and thus

$$[u] \cup 1 = 1 \cup [u] = [u].$$

Definition 2. We shall define *cap product.* For $c \in K_n(\tilde{A})$ and $u \in \mathrm{Hom}_A(K_p(\tilde{A}), A)$ $(n \geq p)$, $c \cap u \in K_{n-p}(\tilde{A})$ is defined such that $c \cap u = \sum_i \lambda_i \sigma_i e^n_{p-1} \dots e_0^{n-p+1} \cdot u(\sigma_i e^n_n \dots e^{p+1}_{p+1})$, where $c \in \sum_i \lambda_i \sigma_i$, $\lambda_i \in A$ and $\sigma_i \in \mathrm{Hom}_A(U^n, A)$.

Furthermore, we define

$$[c] \cap [u] = [c \cap u] \varepsilon H^S_{n-p}(A),$$

which is called the *initial cap product.*

Theorem 3. (i) For $1 \in H^0_S(A)$ and $[c] \in H^S_n(A)$

$$[c] \cap 1 = [c].$$

(ii) For $[c] \in H^S_n(A)$, $[u] \in H^p_S(A)$ and $[v] \in H^{n-p}_S(A)$ $(p \leq n)$

$$[v]([c] \cap [u]) = ([u] \cup [v])([c]).$$

Proof. (i) By Definition 2, for $c = \sum_i \lambda_i \sigma_i$

$$[c] \cap 1 = [c \cap \varepsilon] = [\sum_i \lambda_i \sigma_i] = [c].$$

(ii) By Definition 1 and 2 it suffices to prove that

$$v(c \cap u) = (u \cup v)(c).$$

Put $c = \sum_i \lambda_i \sigma_i$. Since

$$c \cap u = \sum_i \lambda_i \sigma_i e^n_p \dots e_0^{n-p+1} \cdot u(\sigma_i e^n_n \dots e^{p+1}_{p+1}),$$

we have

$$
\begin{aligned}
v(c\cap u) &= \sum_i \lambda_i v(\sigma_i e_p^n \ldots e_0^{n-p+1}) \cdot u(\sigma_i e_n^n \ldots e_{p+1}^{p+1}) \\
&= \sum_i \lambda_i u(\sigma_i e_n^n \ldots e_{p+1}^{p+1}) \cdot v(\sigma_i e_p^n \ldots e_0^{n-p+1}) \\
&= \sum_i \lambda_i (u\cup v)(\sigma_i) \\
&= \sum_i (u\cup v)(\lambda_i \sigma_i) \\
&= (u\cup v)(\sum_i \lambda_i \sigma_i) \\
&= (u\cup v)(c).
\end{aligned}
$$

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