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A Note on the Category Set(H)

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We introduce the concepts of a topological universe and the category **Set(H)** of H-fuzzy sets and show that **Set(H)** satisfies all the conditions of a topological universe except the terminal separator progerty in theorem 2.4.

0. Introduction

The subject of fuzzy sets as an approach to a mathematical representation of vagueness in every day language was introduced by L.A.Zadeh in 1965[14]. He generalized the idea of the characteristic function of a subset of a set X by defining a fuzzy subset of X as'a map from X into [0, 1]. In [4], J.A. Goguen altered this definition to the case in which [0, 1] is replaced by a partially ordered set H.

There are many other categories, for instance, **Set(H)**, $\mathbf{Set}_f$**(H)**, $\mathbf{Set}_g$**(H)** and **Puz(H)** introduced in [3, 4, 9], in connection with fuzzy set theory. However, the category **Set(H)** is the most useful one as the "standard" category, because the category **Set(H)** is very suitable for describing fuzzy sets and maps between them. In particular, the properties of the category **Set(H)** are interesting for their usefulness in connection with the discussion of concepts and also, more essentially, because of the universal nature of the constructions. Until now, many authors have investigated **Set(H)** is topos view-point [1, 2, 3, 4, 13] We will study **Set(H)** in topological uniuerse view-point. The concept of a topological universe was introducd by L.D.Nel [18], which implies a cartesian closed and a concrete quasitopos. The notion of a topological universe has already been put to effective use for several areas of mathematics [11, 12].

In section 1, we introduce basic concepts which are needed in later sections. In section 2, we show that **Set(H)** satisfies all the conditions of a topological universe over **Set** except one for the terminal separator property does not hold in the category.

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1. Preliminaries

In this section, we will introduce some basic definitions which are needed in later section from [4, 5, 6, 7, 8, 10].

Definition 1.1 [7] A lattice H is called a *complete Heyting algebra*, if H satisfies the following conlitions:

(1) H is a complete lattice.

(2) For any $a, b \in H$, the set $\{x \in H \mid x \wedge a \leq b\}$ has a greatest element denoted by $a \rightarrow b$ (called pseudo-complement of a and b), i.e., $x \wedge a \leq b$ if and only if $x \leq (a \rightarrow b)$,

Definition 1.2 [4] Let H be a complete Heyting algebra.

Then the concrete category **Set(H)** is defined by: Objects are (X, ν), called an *H-fuzzy set* (or simply, a *fuzzy set*) on X, where X is any set and ν any map form X to H. A morphism $(X, \nu) \xrightarrow{f} (Y, \eta)$ is a map from X to Y satisfying $\nu(x) \leq \eta \circ f(x)$ for each $x \in X$, where "$\leq$" means the order induced by the operation "$\wedge$ or $\vee$" in H. Every **Set(H)**-morphism will be called a ***Set(H)**-map*.

Definition 1.3 [5] Let **A** be a concrete category whose objects are structured sets, i.e., pairs (X, ξ) where X is a set and ξ is some structure on X (called an *A-structure* on X).

A is said to be *topolgical over* **Set**, if it satisfies the following comdition (The existence of initial structures):

For any set X, any family $((Y_i, \xi_i))_{i \in I}$ of **A**-objects, and any family $(f_i: X \rightarrow Y_i)_{i \in I}$ of maps, there exists an **A**-structure ξ on X satisfying the followings:

(1) (X, ξ) is an **A**-object.

(2) For each $i \in I$, f_i: $(X, \xi) \rightarrow (Y_i, \xi_i)$ is an **A**-morphism.

(3) If (Z, ρ) is an A-object and g: $Z \rightarrow X$ is a map such that for each $i \in Z$, the map $f_i \circ g$: $(Z, \rho) \rightarrow (Y_i, \xi_i)$ is an **A**-morphism, then g: $(Z, \rho) \rightarrow (X, \xi)$ is an **A**-morphism. In this case, ξ is called an *initial* **A**-structure on X with respect to $(X, (f_i)_{i \in I}, ((Y_i, \xi_i))_{i \in I})$ and the family $(f_i$: $(X, \xi) \rightarrow (Y_i, \xi_i))_{i \in I}$ is called an *initial source*.

We can define the concepts of *cotopological categories*, *final structures* and *final sinks* by reversing of the arrow in the concrete category A.

Definition 1.4. [8] Let **A** be a concrete category.

(1) The **A**-*fibre* of a set X is the class of all **A**-structures on X.

(2) **A** is called *properly fibred over* **Set** provided that the following conditions hold:

(i) (*Fibre-smallness*) For each set X, the **A**-fibre of X is a set.

(ii) (*Terminal separator property*) If X is a signleton set, then **A**-fibre of X has precisely one element.

(iii) If ξ and η are **A**-structures on a set X such that 1_x: $(X,\xi)\to(X,\eta)$ and 1_x: $(X,\eta)\to(X,\xi)$ are **A**-morphisms, then $\xi=\eta$.

Definition 1.6 [10] A category **A** is called a *topological universe over* **Set** provided that the following conditions hold:

(1) **A** is well-structured over **Set**, i.e. (i) **A** is a concrete category: (ii) **A** has the fibre smallness condition; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final episinks in **A** are preserved by pullbacks, i.e. for any final episink $(g_\lambda: X_\lambda\to Y)_\Lambda$ and any **A**-morphism f: $W\to Y$, then the family $(e_\lambda: U_\lambda\to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

2. Main Results

Theorem 2.1. The category **Set(H)** is topoloyical over **Set**.

Proof. Let X be any set and $((X_i,\nu_i))_I$ any family of fuzzy sets indexed by a class I. Suppose $(f_i: X\to X_i)_I$ is a source of maps. Define ν: $X\to H$ by $\nu(x)=\bigwedge_I \nu_i\circ f_i(x)$ for each $x\in X$. Then f_i: $(X,\nu)\to(X_i,\nu_i)$ is a **Set(H)**-map for each $i\in I$. For an arbitrary fuzzy set (Y,μ), let g: $Y\to X$ be any map for which $f_i\circ g$: $(Y,\nu)\to(X_i,\nu_i)$ is a **Set(H)**-map for all $i\in I$. Then clearly, $\mu\leq\nu_i\circ(f_i\circ g)-(\nu_i\circ f_i)\circ g$ for all $i\in I$ and hence $\mu\leq(\bigwedge_I\nu_i\circ f_i)\circ g=\nu\circ g$. Thus g: $(Y,\mu)\to(X,\nu)$ is a **Set(H)**-map and hence $(f_i: (X,\nu)\to(X_i,\nu_i))_I$ is an initial source in **Set(H)**. This completes our proof. ///

Examples: (1) *Inverse image of a fuzzy set structure.* Let X be a set, (Y,μ) a fuzzy set, f a mapping of X into Y; the initial fuzzy set structure ν on X for which f: $(X,\nu)\to(Y,\mu)$ is a **Set(H)**-map is called the ***inverse image*** under f of the fuzzy set structure μ of Y. The particular case in which X is a subset of Y and f: $X\to Y$

is the canonical injection; X, with the inverse image μ_x of μ under f, is then called a *fuzzy subset* of (Y,μ). In fact, $\mu_x(x)=\mu(x)$ for all $x\in X$.

(2) *Fuzzy product structure.* Let $((X_i,\nu_i)_I$ be any family of fuzzy sets. The initial fuzzy set structure ν on the product set $X=\Pi X_i$ with respect to $(pr_i)_{i\in I}$ of projections is called the *product* of the fuzzy set structures of the X_i and denoted by $\nu=\Pi\nu_i$ and $(\Pi X_i, \Pi\nu_i)$ is called the *furry product set* of $((X_i,\nu_i))_I$. In fact, $\Pi\nu_i=\bigwedge_I \nu_i\circ pr_i$. In particular, if $I=\{1,2\}$, then $(\nu_1\times\nu_2)(x_1,x_2)=\nu_1(x_1)\wedge\nu_2(x_2)$ for all $(x_1,x_2)\in X_1\times X_2$.

It is well-known [8] that a category is topological if and only if it is cotopological. However, we show directly that **Set(H)** is cotopological.

Theorem 2.2. The category **Set(H)** is cotopological over **Set**.

Proof. Let X be any set and let $((X_i,\nu_i))_I$ be any family of fuzzy sets indexed by a class I. Suppose $(f_i\colon X_i\to X)_I$ is a sink of maps. Define $\nu\colon X\to H$ by for each $x\in X$, $\nu(x)=\bigvee_i \bigvee_{x_i\in f_i^{-1}(x)} \nu_i(x_i)$. Then $f_i\colon (X_i,\nu_i)\to(X,\nu)$ is a **Set(H)**-map. For each fuzzy set (Y,μ), let $g\colon X\to Y$ be any map such that $g\circ f_i\colon (X_i,\nu_i)\to(Y,\mu)$ is a **Set (H)**-map for all $i\in I$. Take any $x\in X$. Then for each $i\in I$ and each $x_i\in f_i^{-1}(x)$, $\nu_i(x_i)\le\mu\circ g\circ f_i(x_i)=\mu\circ g(f_i(x_i))=\mu\circ g(x)$. Thus $\nu(x)\le\mu\circ g(x)$ and hence $g\colon (X,\nu)\to(Y,\mu)$ is a **Set(H)**-map. Therefore **Set(H)** is cotopological over **Set**. ///

Theorem 2.3. Final episinks in **Set(H)** are preserved by pullbacks.

Proof. Let $(g_\lambda\colon (X_\lambda,\nu)_\lambda\to(Y,\eta))_\Lambda$ be any final episink in **Set(H)** and $f\colon(W,\mu)\to(Y,\eta)$ any **Set(H)**-map. For each $\lambda\in\Lambda$, let $U_\lambda=\{(\omega,x_\lambda)\in W\times X_2 \mid f(\omega)=g_\lambda(x_\lambda)\}$ and $\rho_2=(\mu\times\nu_2)\mid U_\lambda\times U_\lambda$, where e_λ and p_λ are the usual projections of U_λ. Then for each $\lambda\in\Lambda$, $e_\lambda\colon(U_\lambda,\rho_\lambda)\to(W,\mu)$ and $p_\lambda\colon(U_\lambda,\rho_\lambda)\to(X_\lambda,\nu_\lambda)$ are **Set(H)**-maps and the following diagram is a pullback square in **Set(H)**:

$$\begin{array}{ccc} (U_\lambda,\rho_\lambda) & \xrightarrow{p_\lambda} & (X_\lambda,\nu_\lambda) \\ {\scriptstyle e_\lambda}\downarrow & & \downarrow{\scriptstyle g_\lambda} \\ (W,\mu) & \xrightarrow[f]{} & (Y,\eta). \end{array}$$

Indeed, $(e_\lambda\colon (U_\lambda,\rho_\lambda)\to(W,\mu))_\Lambda$ is an episink in **Set(H)**: Let $\omega\in W$. Since $(g_\lambda)_\Lambda$ is an episink, there exists $\lambda\in\Lambda$ such that $g_\lambda(x_\lambda)=f(\omega)$ for some $x_\lambda\in X_\lambda$. Thus (ω,x_λ)

$\in U_\lambda$ and $\omega = e_\lambda(\omega, x_\lambda)$. Hence $(e_\lambda)\Lambda$ is an episink. Moreover, $(e_\lambda)\Lambda$ is final: Let μ^* be the final structure on W with respect to $(e_\lambda)\Lambda$. Then for any $\omega \in W$,

$$
\begin{aligned}
\mu(\omega) &= \mu(\omega) \wedge \mu(\omega) \\
&\leq \mu(\omega) \wedge \eta \circ f(\omega) \; (f:(W,\mu) \to (Y,\eta) \text{ is a } \mathbf{Set}(\mathbf{H}\text{-map.})) \\
&= \mu(\omega) \wedge [\bigvee_{\lambda \in \Lambda} \bigvee_{x_\lambda \in g_\lambda^{-1}(f(\omega))} \nu_\lambda(x_\lambda)] \; ((g_\lambda)_\Lambda \text{ is final.}) \\
&= \bigvee_{\lambda \in \Lambda} \bigvee_{x_\lambda \in g_\lambda^{-1}(f(\omega))} [\mu(\omega) \wedge \nu_\lambda(x_\lambda)] \\
&= \bigvee_{\lambda \in \Lambda} \bigvee_{(\omega, x_\lambda) \in e_\lambda^{-1}(\omega)} [\mu(\omega)\nu_\lambda(x_\lambda)]. \\
&= \bigvee_{\lambda \in \Lambda} \bigvee_{(\omega, x_\lambda) \ni e_\lambda^{-1}(\omega)} \rho_\lambda(\omega, x_\lambda).
\end{aligned}
$$

Thus $\mu(\omega) \leq \mu^*(\omega)$ for all $\omega \in W$, i.e, $\mu \leq \mu^*$. On the other hand, since $(e_\lambda:(U_\lambda,\rho_\lambda) \to (W,\mu^*))_\Lambda$ is final, $1_w:(W,\mu^*) \to (W,\mu)$ is a **Set(H)**-map and hence $\mu^* \leq \mu$. Hence $\mu = \mu^*$. This completes our proof. ///

The category **Set(H)** is not properly fibred over **Set**, Since, for any singleton set $\{a\}$, the fuzzy set structure ν on $\{a\}$ is not unique. Hence by theorem 2.1 and 2.3, we obtain the following result.

Theorem 2.4. The category **Set(H)** satisfies all the conditions of a topological universe over **Set(H)** except the terminal separator property.

J.C. Carrega showed [2] that the category **Set(H)** satisfies all the properties of a topos except subobject classifier. He obtained exponential objects in **Set(H)** as follows: For any fuzzy sets (X,ν) and (Y,η), let Y^x be the set of all maps from X into Y and define ρ: $Y^x \to H$ by $\rho(f) = \bigvee \{h \in H \mid \nu(x) \wedge h \leq \eta \circ f(x)$ for all $x \in X\}$. Then (Y^x, ρ) is the exponential object for (X,ν) and (Y,η).

References

1. M. Barr, Fuzzy set theory and topot theory, *Canad. Math. Bull.* **29**(1986)
2. J.C. Carrega, The category Set(H) and Fuz(H), *Fuzzy Sets and Sxstems* **9**(1983), 327~332.
3. M. Eytan, Fuzzy sets; a topos-logical point of view, *Fuzzy Sets and Systems* **5**(1981), 47~67.
4. J.A. Goguen, Categories of V-sets, *Bull. Amer. Math. Soc.*, **75**(1969), 622~624.
5. H. Herrlich, Cartesian closed topological categories, *Math. Coll. Univ. Cape Town* **9**

(1974), 1~16.

6. H. Herrlich and G.E. Strecker, Category theory (Allyn and Bacon, 1973).
7. P.T. Johnstone, Stone spaces, Cambridge University Press 1982.
8. C.Y. Kim, S.S. Hong, Y.H. Hong and P.U. Park, Algebras in cartesian closed topological categories, Lecture Note Series 1. Seminars in Math, Yon Sei Univ., (1979).
9. C.V. Negoita and C.Al. Stefanescu, Fuzzy objects in topoi; a generalization of fuzzy sets, *Bul. Inst. Politehn. Iasi* **24**(1978), 25~28.
10. L.D. Nel, Topological universes and smooth Gelfand-Naimark duality, Mathematical applications of category theory, *Proc. A.M.S. Spec. Session Denver*, 1983, *Contemporary Mathematic* **30**(1984), 224~276.
11. L.D. Nel, Enriched locally conves structures, Differential calculus and Riesz representations, *J. Pure Appl. Algebra* **42**(1986), 165~184.
12. L.D. Nel, Categorical differential calculus for complete locally convex spaces Carleton Univ., Canada KIS 5 B 6 (1987).
13. A.M. Pitts, Fuzzy sets do not form a topos, *Fuzzy Sets and Systems* 8(1982) 101~104.
14. L.A. Zadeh, Fuzzy sets, *Information and Control* 8(1965), 338~353.