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Code Construction Methods for Error Discriminating and Unidirectional Error Control Codes

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Abstract

A new concept, namely the error discrimination of a code defined as the capability to not only detect errors from two distinct error sets but also to distinguish between them has been introduced in [SAKA 89a]. Consider E_+ and E_- as the two distinct error sets, namely the positive error set and the negative error set respectively. If a code C is not only capable of detecting any error e in $\{E_+, E_-\}$, but also able to identify the error set to which e belongs then the code is said to be an E_+ & E_- error discriminating code. The error discriminating property enables construction of unidirectional error detecting/correcting codes using asymmetric error control code.

We derive here theory for asymmetric t error correcting and d error detecting codes. Furthermore, unidirectional error control code construction methods are introduced using asymmetric error control codes and E_+ & E_- error discriminating codes.

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1. Introduction

It is well established that asymmetric error control codes and unidirectional error control codes have useful applications in improving the reliability of communication/computing systems [BOSE 82a, b, 85, 86] [UYEM 87] [RAO 89]. These codes also provide higher information rates than symmetric error control codes [CONS 79] [BOSE 82a, b, 85, 86] [LIN 88] [RAO 89] [DARI 89a] etc.

Considerable research has been done to develop important theories for asymmetric error control codes and unidirectional error control codes. Efficient code construction methods for these codes have also been proposed [KIM 59] [VARS 73] [RAO 75] [CONS 79] [BOSE 82a, b, 85, 86] [DELS 82a, b] [LIN 88] [DARI 89b] etc. However, the theoretical foundation for these codes has not been well established. That is, the neccssary and sufficient condition of t-Asymmetric Error Correcting and simultaneously d-Asymmetric Error Detecting (t-AEC & d-AED) code is not established as yet. Recently, the necessary and sufficient condition t-UEC & d-UED code has been given in [SAKA 82b]. In this paper, we shall develop theory for asymmetric error control codes and show some efficient code construction methods for unidirectional error control codes by using a special *error discriminating code* [SAKA 89a] and asymmetric error control codes.

We review the formal definitions of binary symmetric errors, asymmetric errors and unidirectional errors [BOSE 82a]. In the sequel we will refer to the transition $0 \to 1$ as *0-error* and to the transition $1 \to 0$ as *1-error*.

Symmetric Errors : If both 0-errors and 1-errors appear in a received word with equal probability then the errors are called symmetric type.

Asymmetric Errors : If only one type of errors (0-error type or 1-error type) occur in the received words and the error type is known *a priori* then the errors are called asymmetric type.

Unidirectional Errors : If both 1-errors and 0-errors can occur in the received words, but in any particular word all errors of one type, then they are called unidirectional errors.

The difference between an asymmetric error set and a unidirectional error set is that the direction of error is known *a prior* for the former. Therefore, we can easily observe that a unidirectional error control code which can identify errors into 0-error type and 1-error type, has more error control capacity than that of an asymmetric error control code.

The ***error discrimination capability*** is defined as the capability to distinguish errors between two distinct error sets [SAKA 89a]. Consider E_+ and E_- as the two distinct error sets, the positive error set and the negative error set respectively. An ***E_+ & E_- error discriminating code*** not only detects any error e in $\{E_+, E_-\}$, but also identifies the error set to which e belongs. Note that the error discriminating property enables construction of unidirectional error control codes using asymmetric error control codes.

This paper establishes two important results on the theories for error control codes :

(1) The necessary and sufficient conditions of E_+ & E_- error discriminating codes is the same as the necessary and sufficient condition of AUED or AAED codes.

(2) The necessary and sufficient condition of t-AEC & AAED codes, t-UEC & AUED codes, and t-EC & AUED codes are the same.

Furthermore, we introduce some construction methods for t-UEC codes and t-UEC & AUED codes by adding E_+ & E_- error discriminating codes into the asymmetric error control codes.

2. Preliminary

For our purpose, we define some notations which will be used to determine the capabilities of error discriminating codes, asymmetric error and unidirectional error control codes.

From now on, we use Z as the set of integer, Z^n_2 as a set of binary n-tuples, Z^n_{-2} as a set of n-tuples over $\{0, -1\}$ as $Z^n_{\pm 2}$ as a set of n-tuples over $\{0, \pm 1\}$, Hamming weight and Hamming distance, denoted $W_{Ham}(z)$ and $D_{Ham}(x, y)$ $(\triangleq W_{Ham}(x-y))$ for $x, y, z \in Z^n_2$, have been used to develop the coding theory of symmetric error control codes [PETE 72]. However, for asymmetric and unidirectional error control codes, $W_N(z)$ and $N(x, y)$ $(\triangleq W_N(x-y))$ functions have been used more conveniently to develop the theory [RAO 75] [BOSE 82a]. Recall that $N(x, y)$ is number of $1 \rightarrow 0$ crossovers from x to y, i.e.,

$$N(x, y) \triangleq W_S(x\text{-}y) \triangleq \sum_{i=0} (x_i - y_i)$$

It is well known that the Hamming distance D_{Ham} and the asymmetric distence D_A are obtained as

$$D_{Ham}(x, y) = N(x, y) + N(y, x),$$
$$D_A(x, y) = \max\{N(x, y), N(y, x)\}.$$

In addition to these two weights and distances, *semiweight* $W_S(z)$ and *semidistance* $D_S(x, y)$ $(\triangleq W_S(x - y))$ [RAO 89] are given as

$$D_S(x, y) \triangleq W_S(x\text{-}y) \triangleq \min\{W_S(x\text{-}y), W_S(y\text{-}x)\}.$$

Note that the semidistance is not a metric, since it does not satisfy the triangular inequality property. In the next section, we shall show that D_S is a very important function as it helps in determining the capability of asymmetric codes or unidirectional codes. In this paper, the functions — N, D_{Ham}, D_A, and D_S — will be used to determine the capabilities of error discriminating codes, asymmetric codes, and unidirectional codes.

Throughout this paper, the addition and subtraction of two binary n-tuples is defined by the component-wise addition or subtraction in integer ring Z, i. e.,

$$a \pm b \triangleq (a_0 \pm b_0,\ a_1 \pm b_1,\ a_2 \pm b_2, \cdots, a_{n-1} \pm b_{n-1})$$

where

$$a = (a_0, a_1, a_2, \cdots, a_{n-1}),\ \text{and}\ b = (b_0, b_1, b_2, \cdots, b_{n-1}).$$

Note that the Hamming weight is well defined not only over a finite field but also over the integer ring [PETE 72]. It is also clear that N, D_A and D_S are well defined over the integer ring.

3. Theory

For error control codes, only error correcting and error detecting codes have been considered so far, However, a new error control concept called ***an error discriminating code*** has been introduced in [SAKA 89a]. Here, we consider a special case of error discriminating code called ***E_+ & E_- error discriminating codes*** for the error sets as defined in Definition 3-1.

[Definition 3-1] Positive and Negative Error Set :

$E_+ \triangleq \{e(\neq 0) \mid e \in Z^n_{2}\}$,

$E_- \triangleq \{e(\neq 0) \mid e \in Z^n_{-2}\}$.

[Definition 3-2] : A code C is called an E_+ & E_- Error Discriminating Code *if and only if* the code can discriminate an error e between E_+ and E_-.

□

Based on the above definitions, the necessary and sufficient condition for E_+ & E_- error discriminating codes can be established by Theorem 3-1.

[Theorem 3-1] : A code C is an E_+ & E_- Error Discriminating Code *if and only if*

$$D_S(x, y) \geq 1 \text{ for any distinct } x, y \in C. \quad (3.1)$$

□

Proof : For any $x \in C$, let S_{x+} and S_{x-} denote the sets as follows :

$S_{x+} \triangle \{x+e_+ \mid e_+ \in E_+\}$,

$S_{x-} \triangle \{x+e_- \mid e_- \in E_-\}$.

Then the conditions of E_+ & E_- error discriminating codes are that for any pair x and $y \in C$,

$$[S_{x+} \cap C = \phi] \text{ and } [S_{y-} \cap C = \phi], \text{ and} \quad (3.2)$$

$$S_{x+} \cap S_{y-} = \phi. \quad (3.3)$$

Note that x can be the same as y. Condition (3.2) is for error detection and Condition (3.3) is for error discrimination. For the proof it is required to show that for any distinct $x, y \in C$

Condition (3.1) is equivalent to Condition (3.2) and Condition (3.3).

(i) Condition (3.1) → Condition (3.2) & (3.3) :

Suppose we have c_x, c_y, z such that for some $x, y \in C$

$[S_{x+} \cap C = c_x]$ or $[S_{y-} \cap C = c_y]$, or (3.2')

$S_{x+} \cap S_{y-} = z$. (3.3')

Note that x can be the same as y. Condition (3.2') means that we get $e_+(e_-) \in E_+(E_-)$ such that for some $x, z \in C$.

$x = z + e_+ (\text{or } e_-)$ (3.4)

and (3.1) holds. It is easily observed that (3.4) contradicts Condition (3.1) since for $x, w \in C$, x and w have at least following two columns from Condition (3.1)

$x = (\cdots 0 \cdots 1 \cdots)$,

$w = (\cdots 1 \cdots 0 \cdots)$.

Condition (3.3') means that $e_+(e_-) \in E_+(E_-)$ such that for some $x, y \in C$

$z = x + e_+ = y + e_-$.

Then z covers x and y covers z, and therefore y covers x. This contradicts of condition (3.1).

(ii) Condition (3.2) & (3.3) → Condition (3.1) :

Suppose there are some $x, y \in C$ with $x \neq y$ such that x covers y and Condition (3.2) Condition (3.3) hold. Then we have $e_+ (\in E_+)$ such that

$x = y + e_+$.

That is, $S_{y+} \cap C \neq \phi$. Thus, it is contradiction of condition (3.2) and completes this proof.

(Q.E.D.)

Considerable research has been done for asymmetric error control codes [KIM 59] [VAR 73] [RAO 75, 89] [CONS 79] [DELS 82a, b] [DARI 89b] [SAKA 89c] and unidirectional error control codes [BOSE 28a, b, 85, 86] [LIN 88] [RAO 89] [DARI 89a]. However, theoretical foundation for these codes is not yet well established. We here develop the basic theory for t-AEC & d-AED codes and discuss t-UEC & d-UED codes using t-AEC & d-AED codes and $E_{\pm}$ error discriminating codes.

The conditions for d-asymmetric (or unidirectional) error detecting codes as given in [BOSE 85] are

[Theorem 3-2] [BOSE 85] : A code C is a d-AED code (or d-UED code) *if and only if*

$D_S(x, y) \geq 1$ or $D_{Ham}(x, y) \geq d+1$ for any distinct $x, y \in C$.

□

AAED (AUED) code is a special case of d-AED (dUED) codes with $d=n$. Therefore, the following corollary follows immediately from Theorem 3-2.

[Corollary 3-1] : A code C is AAED code *if and only if*

$$D_S(x, y) \geq 1 \text{ for any distinct } x, y \in C. \quad (3.5)$$

□

Note that the necessary and sufficient conditions for AAED codes, AUED codes, and E_+ & E_- error discriminating codes are the same. For d-error detecting capability, the conditions for asymmetric and unidirectional code are also the same. However, for t-error correcting capability, the conditions for asymmetric codes and unidirectional codes are slightly different and the information rate for asymmetric error control code constructions are better than those for unidirectional error control code constructions.

[Theorem 3-3] [CONS 79] [BOSE 82a] [RAO 89] : For any distinct $x, y \in C$,

(1) t-AEC codes : A code C is a t-AEC code *if and only if*

$D_A(x, y) \geq t+1$,

(2) t-UEC codes : A code C is a t-UEC code *if and only if*

(a) $D_A(x, y) \geq t + 1$ and $D_S(x, y) \neq 0$, or

(b) $D_A(x, y) \geq 2t + 1$ and $D_S(x, y) = 0$.

□

It is easily observed that Condition (a) for t-UEC codes is equivalent to the combination of the condition for t-AEC codes and the condition for E_+ & E_- error discriminating codes.

The necessary and sufficient conditions for t-AEC & d-AED (t-UEC & d-UED) codes are the fundamental conditions for asymmetric (unidirectional) error control codes. Both conditions are defined by the following theorems.

〔Theorem 3-4〕: A code C is a t-AEC & d-AED code *if and only if*

(i) $D_{Ham}(x, y) \geq t + d + 1$ or

(ii) $D_A(x, y) \geq d+1$ or for any distinct $x, y \in C$.

(iii) $D_S(x, y) \geq t+1$

Proof : Without loss of generality we assume 1-error type. The sufficiency of the condition is fairly obvious, since for any distinct $x, y \in C$, Condition (i) is sufficient for t-EC & d-ED codes. Similarly, Condition (ii) is sufficient for d-AEC codes, and Condition (iii) is sufficient for t-EC & AUED codes. That is, any C which has the properties of (3.6) is also a t-AEC & d-AED code.

To prove the necessity of (3.6), let C be a t-AC & d-AED code and assume (3.6) does not hold for C. Then, there exists some distinct $x, y \in C$ such that,

$N(x, y) + N(y, x) \leq t + d$ and

$\max\{N(x, y), N(y, x)\} \leq d$ and

$\min\{N(x, y), N(y, x)\} \leq t$.

We assume without loss of generality, that $N(x, y) \leq t$ & $N(y, x) \leq d$ and some distinct $x, y \in C$ are given as

$$
\begin{array}{l}
\qquad\qquad N(x,y) \qquad N(y,x) \\
x = (\,1\ 1\ \cdots\ 1\ \ 0\ 0\ \cdots\ 0\ 1\ 1\ \cdots\ 1\ \ 0\ 0\ \cdots\ 0\,) \\
y = (\,0\ 0\ \cdots\ 0\ \ 1\ 1\ \cdots\ 1\ 1\ 1\ \cdots\ 1\ \ 0\ 0\ \cdots\ 0\,) \\
z = (\,0\ 0\ \cdots\ 0\ \ 0\ 0\ \cdots\ 0\ 1\ 1\ \cdots\ 1\ \ 0\ 0\ \cdots\ 0\,)
\end{array}
$$

It is then easily observed that there exist some $e_t, e_d \in E_-$ such that $W(e_t) \leq t$, $W(e_d) \leq d$, and

$$z = x + e_t = y + e_d.$$

That is, C can not be an e_d-error detecting code. By a similar argument C cannot be an e_d- error detecting code for errors in E_+. Thus C is not a t-AEC & d-AED code. This is a contradiction and the proof is complete.

(Q. E. D.)

We can easily infer the following corollary from the definitions of asymmetric codes, unidirectional codes and error discrimination codes conditions. Note that if a code C is unordered than for any distinct $x, y \in C$, $D_S(x, y) \geq 1$.

〔Corollary 3-2〕: If a t-AEC & d-AED code C is unordered, then C is also a t-UEC & d-UED code.

〔Theorem 3-5〕〔SAKA 89b〕: A code C is a t-UEC & d-UED code ***if and only if*** for any distinct $x, y \in C$,

(i) $D_{Ham}(x, y) \geq t + d + 1$ or

(ii) $D_A(x, y) \geq d+1$ and $D_S(x, y) \geq 1$ or (3. 7)

(iii) $D_S(x, y) \geq t + 1$.

It is interesting to observe the difference between the condition for t-UEC & d-UED codes and the condition for t-AEC & d-AED codes. The only difference is that Condition (ii) for t-UEC & d-UED codes is more restrictive than Condition (ii) for t-AEC & d-AED codes. That is, a condition of t-UEC & d-UED code is a condition for t-AEC & d-AED code with error discriminating capability.

Note that d-AED(d-UED) codes, t-AEC(t-UEC) codes, and AAED(UAED) codes are special cases of t-AEC & d-AED(t-UEC & d-UED) codes given in Theorem 3-4(3-5) with t=0, t=0 & $d=n$, and $t=d$ respectively. By assigning $d=n$, we also get the necessary and sufficient conditions for t-AEC & AAED(t-UEC & AUED) codes from Theorem 3-4(3-5) as the following corollary shows.

[Corollary 3-3] : A code C is a t-AEC & AAED(t-UEC & AUED) code *if and only if*,

$D_s(x, y) \geq t+1$.   for any distinct $x, y \in C$.

□

Note that the condition for t-AEC & AAED codes, the condition for t-UEC & AUED codes, and the condition for t-EC & AUED codes are the same. From Theorem 3-5, it is also noted that the necessary and sufficient conditions for SUEC & DUED codes are the same as the necessary and sufficient conditions for SEC & DED codes.

4. Code Constructions

4.1 E_+ & E_- Error Discriminating Codes

Recall that the necessary and sufficient condition for E_+ & E_- error discriminating codes is the same as the necessary and sufficient condition for AUED codes. Thus, all AUED codes are also E_+ & E_- error discriminating code. For example, a constant weight code or a Berger code is also an E_+ & E_- error discriminating code.

For the constant code for $m=[n/2]$ or $[n/2]$, the number of codewords is maximized where n is code length and m is the constant weight of a code. The class of $[n/2]$-out-of-n codes has the highest possible information rate among all AUED codes and is therefore called ***an optimal class of AUED codes*** [ANDE 71]. It is also known that Berger codes [BEGR 61] are ***optimal systematic AUED codes*** [BOSE 82a][RAO 89]. Thus, we have an optimal E_+ & E_- error discriminating code as defined by the following corollary.

〔Corollary 4-1〕:

(1) Let n be code length and m be weight. Then a constant code m= [n/2] or [n/2] is an optimal nonsystematic E_+ & E_- error discriminating code.

(2) A Berger code is an optimal systematic E_+ & E_- error discriminating code, in the sense that the information rate k/n is the highest possible for any information block length k.

□

The decoding algorithms of constant codes and Berger codes for E_+ & E_- error discriminating codes are very simple and are shown by means of the following examples.

〔Example 4-1〕 E_+ & E_- error discriminating codes:

Constant codes: Let n be the code length, m be the constant weight, A be transmitted codeword, e be error, and A′(=A+e) be received word of A.

Decoding: 1. $W_{Ham}(A) = W_{Ham}(A')$ → No error,

2. $W_{Ham}(A) > W_{Ham}(A')$ → 1-error(s) ($e \in E_-$),

3. $W_{Ham}(A) < W_{Ham}(A')$ → 0-error(s) ($e \in E_+$),

Let A= (0110) and A′= (0110) + (1000). Since $W_{Ham}(A) < W_{Ham}(A')$, $e \in E_+$.

Berger code: Let B be Berger check for information A, AB be transmitted word, A′B′(=AB+e) be received word of AB, $|A'_0|$ be number of zeros in A′, and D(B) be the decimal representation of B.

Decoding: 1. $D(B') = |A'_0|$ → No error,

2. $D(B') < |A'_0|$ → 1-error(s) ($e \in E_-$),

3. $D(B') > |A'_0|$ → 0-error(s) ($e \in E_+$),

Let AB= (1001001 100) and e = (0010010 001). The A′B′= (1011011 101). Since D(B′) $= 5 > |A'_0| = 2$, $e \in E_+$

4.2 t-UEC Codes

From Theorem 3-5, it is noted that for t-UEC & $d(>t)$-UED code constructions, only t-EC & d-Ed codes, d-UEC codes, or t-EC & AUED codes can be used, and no other cases are possible. That is, (i) and (iii) of Theorem 3-5 are the same as (i) and (iii) of Theorem 3-4. In (ii) of Theorem 3-5, the unordered condition of the code is added to (ii) of Theorem 3-4. We propose a new technique for code construction based on (ii) of Theorem 3-5.

A theorem for t-UEC code construction from t-AEC codes with at most 3 appending bits has been shown in [DARI 89a]. The t-UEC codes constructed by the theorem in [DARI 89a] may suffer from a lack of efficient encoding/decoding algorithms. We here show a t-UEC code construction method with simple encoding/decoding algorithm. The code construction method is based on Corollary 3-2.

[Encoding Algorithm] :

Step 1 : Encode an t-AEC code C_A.

Step 2 : Induce a partition

$$\{P_0, P_1, \cdots, P_{m-1}\} \text{ on } C_A \tag{4.1}$$

such that two codewords x and y are in the same partition if and only if $W_{Ham}(x) = W_{Ham}(y)$, and $W_{Ham}(x) > W_{Ham}(z)$ if and only if $x \in P_i$, $z \in P_j$ and $i > j$.

Step 3 : Let $S_D \triangle \{W_N(x) \mid x \in C_A\}$ and $S_R \triangle \{0, 1, \cdots, m\text{-}1\}$ where m is the number of partition. Define two function ν and ϕ as follows :

$$\nu(W_N(u)) = i, \quad : S_D \rightarrow S_R \text{ (one-to-one and onto mapping)}$$

$$\phi(i) = (m\text{-}i\text{-}1)_2 : S_R \rightarrow Z_2^{\beta} \text{ (into mapping)}$$

where $\beta = \lceil \log_2 m \rceil$ and $(j)_2$ is a binary representation of $j \in Z$.

Step 4 : Construct the code C_U as follows :

$$C_U \triangleq \{c = (u, b) \mid b \triangleq \phi \cdot \nu(W_N(u)), \ u \in C_A\}.$$

□

Each partition of (4.1) is a constant coe. That is, each partition is guaranteed to be a E_t error discriminating code. It is also guaranteed that $D_S(x\text{-}y) \geq 1$ for all $x \in P_i$ and $y \in P_j$

for $i > j$. Moreover, for the modified Berger checks b_i for P_i and b_j for P_j, $D_s(b_j - b_i \geq 1$; since $b_j > b_i$ for $i > j$.

〔Corollary 4-2〕: The proposed code C_U constructed by above encoding method is t-UEC code

□

The decoding algorithm for this construction is also simple and given as follows:

〔Decoding algorithm〕:

Step 1 : Find the error directions using the mapping ϕ. Let $c = xb$ be transmitted codeword and $c' = x'b'$ be received word where $b = \nu(\phi(D(x)))$.

(i) No unidirectional errors : $W_N(x') = \nu^{-1}\phi^{-1}(b')$,

(ii) Positive errors :

1. $\phi^{-1}(b')$ is not well defined and $W_N(b) > m-1$ where $b = (b_0, b_1, \cdots, b_{n-l})$.
2. $\phi^{-1}(b')$ is well defined and $W_N(b) > \nu^{-1}(\phi^{-1}(b'))$.

(iii) Negative errors :

1. $\phi^{-1}(b')$ is not well defined and $b_i < 0$ for some i
2. $\phi^{-1}(b')$ is well defined and $W_N(b) < \nu^{-1}(\phi^{-1}(b'))$.

Step 2 : Apply the decoding algorithm of the base asymmetric code with the known error direction.

□

The systematic generation of the t-UEC code depends upon the base t-AEC code. If the t-AEC code is a systematic code, then the constructed t-UEC code is also systematic. The efficiency of t-UEC code construction is very dependent on t-AEC code construction method. Also, the less weight distributions of the base t-AEC codes provide the better information rate of the proposed t-UEC codes.

Several SAEC and t-AEC code construction methods have been shown 〔CONS 79〕〔DELS 81a, b〕〔SHIO 82a, b〕〔CUNN 82〕〔DARI 89b〕〔SAKA 89c〕. These codes provide higher information rates than symmetric error control codes. But further research on more efficient systematic t-AEC code construction methods based on this method would be required.

4.3 t-UEC & AUED codes

We review some t-EC & AUED code construction methods and discuss an efficient systematic t-UEc & AUED code construction procedure based on t-EC & AUED code construction methods. Recall that the condition for t-AEC & AAED codes, the condition for t-UEC & AUED codes, and the condition for t-EC & AAED codes, t-UEC & AUED codes, and t-EC & AUED codes will be the same.

The three basic approaches to systematic t-EC & AUED code construction are given in [BOSE 82b], [NIKO 86], and [KUND 89]. The format for the codewords is the following form shown in Fig. 4. 1.

Information bits	(T)	(U)

Fig. 4. 1 t- EC & AUED code

The systematic t-EC code is used as a base code for the format. The (T) field consists of check bits for the t-EC code. For AUED purposes, (U) check bits are added to the t-EC code.

In [BOSE 82b] the (U) field consists of a t-level Berger check, while the (U) field in [NIKO 86] uses a modified t-level Berger check. In [KUND 89] the (U) field consists of a nonsystematic t-EC & AUED code. In decoding the codes, all of them use (T) for correct t symmetric errors and use (U) for detecting unidirectional errors which are greater than $t+1$.

Since the construction method proposed in [KUND 89] provides the best information rate among the three techniques, we propose a t-UEC & AUED code construction method which is essentially a modification of the [KUND 89] t-EC & AUED code construction. When $t=1$, the code constructed by this method is *asymptotically optimal*. The proposed codeword has the format of Fig. 4. 2.

Systematic t-UEC & AUED code C		
Systematic t-AEC code24(C_A)		Nonsystematic t-UEC & AUED code (C_N)
Information bits (I)	Check bits for t-AEC code (A)	Check bits for AUED (U)

Fig. 4.2 The Format of the Proposed Code.

The base code for the proposed code is a t-AEC code instead of a t-EC code. The (A) field represents the check bits required to make (IA) a codeword in a systematic t-AEC code. A nonsystematic t-UEC & AUED code is used for the (U) field instead of a nonsystematic t-EC & AUED code. The procedure for code construction is similar to that given in [KUND 89] and is briefly outlined below.

[Encoding Algorithm]

Step 1 : Encode a systematic t-AEC code by attaching (A) check bits to (I).

Step 2 : Induce a partition $\{P_1, P_2, \cdots, P_m\}$ on C_A such that two codewords x and y are in the same partiton *if and only if* their weight are identical. The number of blocks in the induced partition (m) is equal to the number of distinct Hamming weights of codewords in C_A.

Step 3 : Let (U) be a t-UEC & AUED code with $\geq m$ codewords. Let f be an arbitrary one-to-one mapping from partition $P = \{P_1, P_2 \cdots, P_m\}$ to C_N. Then the check bits (U) in a codeword C of the t-UEC & AUEd code being constructed are $f(P_i)$ where P_i is the block that contains $(IA) \in C_A$.

□

The decoding algorithm is though slightly different from [KUND 89], follows the same basic principle.

Let $\overline{x_N}$ be the received word for $x_N \in C_N$ and $\overline{x_A}$ be the received word for $x_N \in C_A$.

〔Decoding Algorithm〕

Step 1 : Decode $\overline{x_N}$ using the decoding procedure for C_N and find the error direction.

Step 2 : Decode $\overline{x_A}$ using the decoding procedure for C_A.

Step 3 : If an uncorrectable error is detected in decoding in Step 1 or Step 2, declare that an uncorrectable error-pattern was detected in $\overline{x_A x_N} \in C$ and stop further decoding. Otherwise, let the decoded word be $x_A' x_N'$.

Step 4 : Compare the weight of x_A' with the value implied by x_N'.

Step 5 : If the comparison indicates that the weight of x_A' is the same as that implied by x_N', declare (I) as the transmitted information word. Otherwise, declare an uncorrectable error pattern.

□

The proposed code has the following capability.

〔Theorem 4-2〕: The proposed code of Fig. 4.3 is a t-UEC & AUED code.

The proof is given in the Appendix. □

The above t-UEC & AUED code construction method uses the t-AEC code as a base code instead of a t-EC which is used a base code for the t-EC & AUED in 〔KUND 89〕. The t-UEC & AUED code is attached to the base code instead of the t-EC & AUED code which is attached for the t-EC & AUED in 〔KUND 89〕. Based on the above observation we get the following corollary.

〔Corollary 4-3〕: Let C_{UD} be a t-UEC & AUED code derived by above proposed method and C_{SU} be the t-EC & AUED code derived by the method in 〔KUND 89〕. Then there always exists a C_{UD} whose information rate is equal to or better than that of C_{SU}.

□

Since a constant weight code C, with minimum Hamming distance ($d_{min,Ham}$) $2t+2$ is a t-EC & AUED code 〔RAO 89〕, we obtain the following corollary.

〔Corollary 4-4〕: A constant weight code C with $d_{min,Ham}=2t+2$ is a t-UEc & AUEd code.

[Example 4-4] SUEC & AUED code for k=5 :

Encoding :

Step 1 : Encode the systematic SAEC (8, 5) code by the method shown in [CUNN 82]. The code table is shown in Table 1 in Appendix.

Step 2 : Determine the number of distinct weights for the codewords in C_A. for our example, the number of distinct weight is 7 (see Table 1 in Appendix).

Step 3 : Pick a nonsystematic SUEC/AUED code C_N with 7 codewords. From Corollary 4-4, a constant weight code C with $d_{min,Ham}=4$ is SUEc & AUED code. One such code with 7 codewords has length 7 from the constant weight code table in [RAO 89]. The mapping table for x_A and x_N is given in Table 4.1.

Table 4.1 Mapping Table for x_A and x_N

$W_{Ham}(x_A)$	x_N	$W_{Ham}(x_A x_N)$
0	1 1 1 0 0 0 0	3
2	1 0 0 1 1 0 0	5
3	1 0 0 1 1 0 0	6
4	1 0 0 0 0 1 1	7
5	0 1 0 0 1 0 1	8
6	0 0 1 1 0 0 1	9
7	0 0 1 0 1 1 0	10

If I= (01101) then x_A= (01101 010) from Table 1 in the Appendix and we have $W_{Ham}(x_A)=4$. Hence, x_N= (1000011). Therefore,

$$x = x_A x_N = (01101\ 010\ 1000011). \tag{4.4}$$

Decoding :

Case 1 : Let the transmitted word x be the same as (4.4) and let the error be e= (10000 001 0100000). Then the received word is

$$\bar{x} = (11101\ 011\ 1100011).$$

By decoding $\overline{x_N}$, we get x_N' = (1000011) and the direction of errors is seen to be positive. With the known error direction, we decode $\overline{x_A}$. In this case an uncorrectable error pattern has occurred.

Case 2 : Let the transmitted word x be (00000) and let the error be $e = (11000\ 000\ 0000000)$. The the received word is

$$\bar{x} = (11000\ 000\ 1110000)$$

From Table 4.1, we know the error direction is positive. With the known error direction, we decode $\overline{x_A}$ and from Table 1 in the Appendix we get

$$x_A' = (11100\ 000\ 1110000).$$

However, $W_{H..}(x_A' x_N) = 6$ is not the same as the weight for $x_N = 111000$ (which is 3). Therefore, we declare that the received word is an uncorrectable error pattern.

□

5. Conclusion

In this paper, we developed the necessary and sufficient conditions for E_+ & E_- error discriminating codes and t-AEC & d-AED codes. It has been observed that the necessary and sufficient condition for E_+ & E_- error discriminating codes is the same as the necessary and sufficient condition for AUED, and AAED codes. It has also been observed that the condition for t-AEC & AAED codes, the condition for t-UEC & AUED codes, and the condition for t-EC & AUED codes are the same. Thus, the optimal number of codewords for t-AEC & AAED codes, t-UEC & AUED codes and t-EC & AAED codes will be the same.

We have also developed the code construction methods for t-UEC codes and t-UEC & AUED codes. It has been proved that the information rate of the code derived by the proposed t-UEC & AUED code construction methods is equal to or better than those of t-EC & d-UED code construction method which use specific formats.

However, further research on more efficient systematic t-AEC code construction methods based on the proposed t-UEC & d-UED code construction methods would be beneficial as the efficiency of the proposed construction method are very dependent on the efficiency of t-AEC code construction methods.

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Appendix

Proof of Theorem 4-2: Since the proposed t-UEC & AUED code construction method is based on the t-EC & AUED code construction method in 〔KUND 89〕, the following proof is somewhat similar to the proof outlined in 〔KUND 89〕.

We prove this theorem by showing that the Decoding Algorithm corrects t or less unidirectional errors, detects $t+1$ or more errors and discriminates the direction of errors in the received word.

(1) E_+ & E_- error discriminating capability : The following cases are for all possible error locations.

(i) e in C_N : Since C_N is an AUED code, it is clear that the code can discriminate the errors between E_+ and E_-.

(ii) e in C_A and e in C_N : The direction of error in C_A is the same as that in C_N. Therefore, it is clear that the code can discriminate the errors between E_+ and E_-.

(iii) e in C_A : Given $x_N \in C_N$, the weight of corresponding $x_A \in C_A$ is fixed by the mapping f (see Encoding Step 3). Therefore, the code can discriminate erros between E_+ and E_-.

(2) t-UEC capability : In (1) above, it is proved that the proposed code is an E_+ & E_- error discriminating code. Since the direction of erros is known, C_A is a t-UEC code. C_N is also a t-UEC code. Therefore, the proposed code can correct upto t unidirectional errors.

(3) $t+1$ or more UED capability : if $\overline{x_N}$ has more than t unidirectional erros, an uncorrectable error pattern is declared (see Decoding Step 3). If $\overline{x_N}$ has less than or equal to t unidirectional errors, then we can get correct $\overline{x_N}$. Let us assume that $\overline{x_A}$ has $t+1$ or more unidirectional errors. Then the weight of $\overline{x_A}$ would be different from the weight of x_A by $t+1$ or more. Note that C_A only can correct upto t unidirectional errors. Hence, the decoded x_A' can only differ in at most t positions from $\overline{x_A}$. Therefore, the weight of x_A' and $\overline{x_A}$ can not be the same. Thus, the corresponding x_N will be different and the received word will be declared as an uncorrectable unidirectional error pattern by Step 5 of Decoding Algorithm.

(Q. E. D.)

Table 1 Table for Systematic SAEC Codes [CUNN 82]

Decimal	Information bits (I)	Check bits (A)	W_{Ham}(IA)
0	0 0 0 0 0	0 0 0	0
1	0 0 0 0 1	0 0 1	2
2	0 0 0 1 0	0 1 0	2
3	0 0 0 1 1	0 1 1	4
4	0 0 1 0 0	0 1 1	3
5	0 0 1 0 1	1 0 0	3
6	0 0 1 1 0	1 0 1	4
7	0 0 1 1 1	1 1 0	5
8	0 1 0 0 0	1 0 0	2
9	0 1 0 0 1	1 0 1	4
10	0 1 0 1 0	1 1 0	4
11	0 1 0 1 1	1 1 1	6
12	0 1 1 0 0	1 1 1	5
13	0 1 1 0 1	0 1 0	4
14	0 1 1 1 0	0 0 0	3
15	0 1 1 1 1	0 0 1	5
16	1 0 0 0 0	1 0 1	3
17	1 0 0 0 1	1 1 0	4
18	1 0 0 1 0	1 1 1	5
19	1 0 0 1 1	0 0 0	3
20	1 0 1 0 0	0 0 0	2
21	1 0 1 0 1	0 0 1	4
22	1 0 1 1 0	0 1 0	4
23	1 0 1 1 1	0 1 1	6
24	1 1 0 0 0	1 0 1	4
25	1 1 0 0 1	1 1 0	5
26	1 1 0 1 0	1 1 1	6
27	1 1 0 1 1	0 0 0	4
28	1 1 1 0 0	0 0 0	3
29	1 1 1 0 1	0 0 1	5
30	1 1 1 1 0	0 1 0	5
31	1 1 1 1 1	0 1 1	7