V-RINGS DETERMINED BY POLYNOMIAL RINGS

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Let $R$ be a ring with identity. A nonzero left $R$-module $M$ is called irreducible (or simple) if $0$ and $M$ are the only $R$-submodules.

If $R$ is a division ring, then $R[x]$ is a principal left and right ideal domain. For an irreducible right $R[x]$-module $M, M=R[x]/I$ as a right $R[x]$-module and $I$ is a maximal right ideal of $R[x]$. But since $R[x]$ is a principal ideal domain, there is a monic polynomial $f(x)$ in $R[x]$ such that $I=f(x)R[x]$ Now say $\deg f(x) = n$. Then the irreducible $R[x]$-module $M=R[x]/I$ is generated by $1+I, x+I, \cdots, x^{n-1}+I$ over $R$ as an $R$-module.

By this well-known standard fact, it is quite natural to raise the following question at least when $R$ is a simple.

**Question 1.** Suppose $R$ is a simple ring. Then is every irreducible right $R[x]$-module finitely generated $R$-module?

But the following example by Resco [7, Example 3.3] nullifies our hope for the affirmative answer for the above question.

**Example 2** [Resco]. Let $k$ be a field of characteristic zero and let $K=k[[x]]$ be the ring of power series over $k$. Then $K$ is a domain. Now let $L=k((x))$ be the field of fractions of $K$. Let $d:K\to k$ be ordinary differentiation and extended $d$ to $L$ in the usual manner. Let $A=K[y,d]$ the differential operator ring over $K$. That is, $A$ is free left $K$-module with basis $\{1, y, y^2, \cdots\}$ and with multiplication extended from $K$ via $ya=ay+d(a)$ for every $a$ in $K$. Also $B=L[y,d]$. Define a $A[x]$-module structure on $B$; for $b$ in $B$ and $f=\sum x^ia$, in $A[x]$, define
Then $B$ is a right $A[x]$-module. Moreover $B$ is an irreducible faithful $A[x]$-module. But $B$, as a right $A$-module, is not finitely generated.

But despite of the above pathological example, we are able to give an affirmative answer to our Question 1 in some situation. Indeed we observe several cases for which every irreducible $R[x]$-module is a finitely generated $R$-module.

**Definition 3.** For a ring $R$, a right $R$-module $M$ is called **bounded** if the annihilator $\text{Ann}_R(M)$ of $M$ in $R$ is nonzero.

For example of bounded modules let $D$ be a division ring with the center field $F$. As we have known that, if $D$ is not purely transcendental over $F$, then the polynomial ring $D[x]$ over $D$ is not primitive. So in this case every irreducible $D[x]$-module is bounded. For, if it were not, then there would exist an irreducible $D[x]$-module $M$ with $\text{Ann}_{D[x]}(M) = 0$. Thus $M$ is a faithful $D[x]$-module and so $D[x]$ is primitive. But this is a contradiction.

The following lemma is a well-known standard fact.

**Lemma 4 [Bergman].** Let $S$ be a finite centralizing ring extension of a ring $R$. If $M$ is an irreducible $S$-module, then as a $R$-module, $M$ is a finite direct sum of irreducible $R$-modules.

The following theorem gives an affirmative answer to Question 1 in some circumstance.

We recall that an overring $S$ of a ring $R$ with same identity is called a **centralizing extension** of $R$ with **finite basis** if $S$ is finitely generated as an $R$-module with centralizing finite basis \{\(u_1, u_2, \ldots, u_n\)\}, that is,

\[
S = \sum_{i=1}^{n} ru_i \text{ with } ru_i = u_ir \text{ for each } r \text{ in } R, \quad i = 1, 2, \ldots, n
\]

and if \(r_1u_1 + r_2u_2 + \cdots + r_nu_n = 0\), then \(r_1 = r_2 = \cdots = r_n = 0\).

**Theorem 5.** For a simple ring $R$ any bounded irreducible right
$R[x]$-module is a finite direct sum of irreducible $R$-module. So it is finitely generated as a right $R$-module. Moreover, any bounded irreducible right $R[x]$-module never be a projective $R$-module unless $R$ is right Artinian.

Proof. Let $M$ be a bounded irreducible right $R[x]$-module. Then there is a maximal right ideal $I$ of $R[x]$ such that $M$ is $R[x]$-module isomorphic to $R[x]/I$. Now since $M$ is a bounded $R[x]$-module, $A=\text{Ann}_{R[x]}(M)$ is a nonzero module of $R[x]$ contained in $I$. Pick

$$g(x)=a_0+a_1x+\cdots+a_nx^n$$

a nonzero polynomial in $A$ with the least degree. Then since we may assume $a_n\neq 0$, $R_aR$ is a nonzero ideal of $R$. By our assumption since $R$ is simple, $R_aR=R$ so we have

$$1=c_1a_1d_1+c_2a_2d_2+\cdots+c_ka_kd_k$$

for some $a_i$ and $d_i$ in $R$, $i=1,2,\ldots,k$. Thus

$$f(x)=c_1g(x)d_1+c_2g(x)d_2+\cdots+c_kg(x)d_k$$

is a monic polynomial in $A$ with the least degree. Now for any $r$ in $R$, $rf(x)\equiv f(x)\equiv f(x)r$ for any $r$ in $R$. Hence $f(x)$ is a monic central polynomial in $R[x]$. In this case we may use the division algorithm with $f(x)$ and get $A=f(x)R[x]$ since $f(x)$ is monic central. Now if we denote $F$ the center of $R$, then $F$ is a field and

$$R[x]/A=R\otimes_FF[x]/f[x]F[x]$$

But since $F$ is a field, as a vector space over $F, F[x]/f[x]F[x]$ is generated by $1+f(x)/f[x], x+f(x)/f[x], \ldots, x^{n-1}+f(x)/f[x]$, where $n$ is the degree of the polynomial $f(x)$. Therefore $1+f(x)R[x], x+f(x)R[x], \ldots, x^{n-1}+f(x)R[x]$, i.e., $1+A, x+A, \ldots, x^{n-1}+A$ is a finite centralizing element of $R[x]/A$ over the ring $R$. Now since $M$ is an irreducible $R[x]$-module and $A=\text{Ann}_{R[X]}M$, considering $M$ as an $R[x]/A$-module it is also an irreducible $R[x]/A$-module. But since $R[x]/A$ is a finite centralizing extension of $R$, irreducible right $R[x]/A$-module $M$ is a finite direct sum of irreducible right $R$-module by Lemma 4. Therefore this $R[x]$-module $M$ is finitely generated as an $R$-module.
Moreover, if $M$ is a projective $R$-module considering as a right $R$-module, then its irreducible $R$-direct summand is also a projective $R$-module. So the $\text{Socle}(R)$ the sum of the minimal right ideal of $R$ is nonzero because any minimal right ideal of $R$ is $R$-isomorphic to an irreducible $R$-direct summand of $M$. But since $R$ is simple, $\text{Soc}(R)=R$. Thus $R$ is simple Artinian. Therefore a bounded irreducible right $R[x]$-module never be a $R$-projective module unless $R$ is Artinian.

**Corollary 6.** Let $R$ be a simple ring. Then if the polynomial ring $R[x]$ is not primitive, any irreducible right $R[x]$-module is a finite direct sum of irreducible $R$-modules.

*Proof* Since the polynomial ring $R[x]$ is not primitive, any irreducible right $R[x]$-module should be bounded. Therefore it follows immediately by the proof of Theorem 5.

By the hint of Theorem 5 we are able to come to the consideration of ring $R$ such that every irreducible $R[x]$-module is a projective $R$-module. The forthcoming result may characterize semi-simple Artinian ring in a new way via $R[x]$-module structure. Roughly speaking, the $R$-projective module property of irreducible $R[x]$-modules influences very strongly on the ring structure of $R$ so that the ring $R$ becomes semisimple Artinian ring.

**Theorem 7.** The followings are equivalent.
( i ) Every irreducible right $R[x]$-module is a projective right $R$-module.
( ii ) $R$ is semisimple Artinian.

*Proof* ( ii ) implies ( i ). Suppose $R$ is semi-simple Artinian. Then any right $R$-module is projective. So it is obvious.

( i ) implies ( ii ). Assume that every irreducible $R[x]$-module is a projective right $R$-module. Define a map $\theta$ from $R[x]$ to $R$ by $\theta(f(x))=f(0)$ for $f(x)$ in $R[x]$. Then $\theta$ is a ring epimorphism. We claim that every irreducible $R$-module is a projective $R$-module. Now for an irreducible right $R$-module $R/B$ with $B$ a maximal right ideal $B$ of $R$, let

$$I=\{f(x)\in R[x] \mid \theta(f(x))\in B\}.$$
Then since $B$ is a right ideal of $R$, $I$ is a right ideal to $R[x]$ and it contains a two-sided ideal $xR[x]$ of $R[x]$.

Pick $f(x) \in R[x] - I$. Then $f(0) \in R - B$ and so $f(0) + B$ is a generator of $R/B$ i.e., $(f(0) + B)R = R/B$. In this case $R[x]/I$ is generated by $f(x) + I$. For, if $g(x) + I$ is in $R[x]/I$, then $g(0) + B = (f(0)r + B$ for some $r$ in $R$. Hence $g(0) - f(0)r$ is in $B$ and therefore $g(x) - f(x)r$ in $I$. So $g(x) + I = (f(x) + I)f = f(x)r + I$. This means that $R[x]/I$ is an irreducible $R[x]$-module.

Now as $R$-modules, $R[x]/I$ is isomorphic to $R/B$. For if we define $\tilde{g}: R[x]/I \to R/B$ by $\tilde{g}(f(x) + I) = f(0) + B$, then it can be straightforwardly checked that $\tilde{g}$ is an $R$-isomorphism. So $R[x]/I$ is a cyclic $R$-module. Now by assumption, since $R[x]/I$ is a projective $R$-module and so is $R/B$. Therefore every irreducible $R$-module is projective. Hence every maximal right ideal is an $R$-direct summand of $R$ because the exact sequence

$$0 \to B \to R \to R/B \to 0$$

do $R$-module with $B$ a maximal right ideal of $R$ is splitted.

Finally to finish our proof, let $J$ be a right ideal of $R$ and let $K$ be a maximal complement of $J$ in $R$, that is, $K$ is a right ideal maximal with respect to the property $J \cap K = O$. Actually, the existence of maximal complement is assured by Zorn's lemma. Then $J + K = J \oplus K$ is an essential right ideal of $R$. If $J \oplus K \subseteq R$, then there is a maximal right ideal $M$ of $R$ such that $J \oplus K \subseteq M \subseteq R$. In this case $M$ is essential since $J \oplus K$ is essential. But by our result in the previous paragraph $M$ is a direct summand of $R$. This is impossible and so $J \oplus K = R$. Therefore every right ideal $J$ of $R$ is an $R$-direct summand of $R$. Hence $R$ is semi-simple Artinian ring.

As in the proof of the above theorem a ring whose every irreducible $R$-module is projective is simple Artinian. With this fact the following definition may be of interest.

**Definition 8.** A ring $R$ is called a right $V$-ring if every right irreducible $R$-module is injective.

In the process of the proof for Theorem 7, we get following.

**Theorem 9.** If every irreducible right $R[x]$-module is an injective $R$-module, then $R$ is a right $V$-ring.
By this result we may consider its converse.

**Lemma 10** [Armendariz and Fisher]. Let \( R \) be a P.I.-ring. Then \( R \) is a von Neumann ring if and only if \( R \) is a V-ring.

**Lemma 11** [Posner]. Let \( R \) be a prime P.I.-ring. Then \( R \) has the classical right quotient ring \( Q(R) \) which is simple Artinian. Also in this case \( Q(R) \) is the classical left quotient ring and \( Q(R) = RF \), where \( F \) is the center of the simple Artinian ring \( Q(R) \). Moreover the field \( F \) is a field of fraction of the domain \( Z(R) \).

**Lemma 12.** Every prime, von Neumann regular P.I.-ring is simple Artinian.

*Proof* Let \( R \) be a prime, von Neumann regular P.I.-ring. Since \( R \) is a von Neumann regular ring, then so is \( Z(R) \). Now let \( 0 \neq a \in Z(R) \), then there is \( b \) in \( Z(R) \) such that \( a = aba \) But since \( R \) is prime, so is \( Z(R) \) and hence \( Z(R) \) is a commutative domain. From the fact \( a = aba \), we have \( a(1 - ba) = 0 \) in \( Z(R) \). So \( 1 = ba \), since \( a \neq 0 \). Thus \( Z(R) \) is a field. So the center \( F \) of \( Q(R) \) is \( Z(R) \) by Lemma 11. Thus \( Q(R) = RF = RZ(R) = R \) and so \( R \) is simple Artinian by Lemma 11.

**Lemma 13.** Let \( R \) be a P.I.-ring. Then the followings are equivalent.

(i) Every irreducible right \( R[x] \)-module is finitely generated over \( R \) as a module.

(ii) Every primitive factor ring of \( R[x] \) is finitely generated over \( R \) as a module.

*Proof* (i) implies (ii) Suppose every irreducible \( R[x] \)-module is a finitely generated \( R \)-module. Let \( A \) be a two-sided ideal of \( R[x] \) such that \( R[x]/A \) is a primitive ring. Then the ring \( R[x]/A \) is a primitive P.I.-ring and so it is simple Artinian by Kaplansky. So the ring \( R[x]/A \) has a minimal right ideal \( I_0 \). In this case \( I_0 \) has the form \( I/A \) with \( A \subseteq I \) and \( I \) is a right ideal of \( R[x] \). We claim that \( I/A \) is an irreducible \( R[x] \)-module. By the module structure defined by

\[(f(x) + A)g(x) = f(x)g(x) + A\]

for \( f(x) + A \) in \( I/A \) and \( g(x) \) in \( R[x] \), \( I/A \) is a right \( R[x] \)-module
compatible with the original module structure of $I/A$ as $R[x]/A$-module. By this newly induced module structure on $I/A$, since $I/A$ is an irreducible $R[x]/A$-module, $I/A$ is an irreducible $R[x]$-module.

Finally, since

$$R[x]/A \cong \bigoplus I_0$$

as $R[x]$-module and $I_0$ is finitely generated as $R$-module by assumption, we have that $R[x]/A$ also is finitely generated as an $R$-module.

(ii) implies (i) Suppose every primitive factor ring of $R[x]$ is finitely generated as an $R$-module. Now let $M=R[x]/N$ be an irreducible $R[x]$-module with $N$ a maximal right ideal of $R[x]$. If $A=\text{Ann}_{R[x]}(M)$, then $R[x]/N$ is a faithful irreducible $R[x]/A$-module. Hence $R[x]/A$ is a primitive P.I.-ring. do $R[x]/A$ is simple Artinian by Kapiasny and hence $A=0$. Now as $R[x]/A$-module we have

$$R[x]/A \cong \bigoplus R[x]/N$$

since $R[x]/A$ is simple Artinian and $R[x]/N$ is an irreducible $R[x]/A$-module. So $M=R[x]/N$ is a finitely generated $R$-module because $R[x]/A$ is a finitely generated $R$-module.

As a byproduct of the above lemma we have the following.

**Proposition 14.** Let $R$ be a P.I. ring. Then the followings are equivalent.
(i) Every irreducible $R[x]$-module is a finitely generated $R$-module.
(ii) Every maximal ideal of $R[x]$ can be contracted to a maximal ideal of $R$.

**Proof** (i) implies (ii) Let $A$ be a maximal ideal of $R[x]$. Then $R/A \cap R \subseteq R[x]/A$ and $R[x]/A$ is simple P.I. So $R[x]/A$ is simple Artinian. By Lemma 13, $R[x]/A$ is a finitely generated $R$-module. Therefore $R[x]/A$ is a finitely generated $R/A \cap R$-module, Hence $R[x]/A$ is a finite centralizing extension of $R/A \cap R$. Hence $R/A \cap R$ is Artinian, since $R[x]/A$ is Artinian. Now since $R/A \cap R$ is prime P.I., it is simple Artinian. So $A \cap R$ is maximal in $R$.

(ii) implies (i) Let $A$ be a primitive ideal of $R[x]$. Then since $R[x]$ is P.I., $A$ is a maximal ideal. So by our assumption, $A \cap R$ is
a maximal ideal of $R$. Hence $R/A \cap R$ is simple P.I. and so it is simple Artinian. Therefore

$$R[x]/A \cong \frac{(R/A \cap R)[x]}{A/(A \cap R)[x]}$$

is a finitely generated $R/A \cap R$-module because $R/A \cap R$ is simple Artinian. Hence $R[x]/A$ is a finitely generated $R$-module. Thus by Lemma 13, we get our conclusion.

Now we are in the situation to characterize $V$-ring whenever it satisfies a polynomial identity.

**Theorem 15.** Let $R$ be a P.I.-ring. Then the followings are equivalent,

(i) Every irreducible right $R[x]$-module is an injective $R$-module.

(ii) $R$ is a (right) $V$-ring.

**Proof.** By Theorem 9, (i) implies (ii) immediately. Now suppose $R$ is a (right) $V$-ring. Then by Lemma 10, $R$ is a von Neumann regular ring. For an irreducible right $R[x]$-module $M$, let $\text{ Ann }_R(M)$ the annihilator of $M$ in $R[x]$. Then the ring $R[x]/A$ has $M$ as a faithful irreducible module. So $R[x]/A$ is a primitive ring. Our claim is that the subring $R+A/\text{ Ann }_R(M)$ of $R[x]/A$ is a prime ring. For this, let $\bar{U}$ and $\bar{V}$ be ideals of $R/\text{ Ann }_R(M)$ such that $\bar{U}\bar{V}=\bar{0}$. Then there are ideals $U,V$ of $R$ such that $\bar{U}=U+\text{ Ann }_R(M)$ and $\bar{V}=V+\text{ Ann }_R(M)$. Then of course $\bar{U}=U+\text{ Ann }_R(M)$ and $\bar{V}=V+\text{ Ann }_R(M)$. Thus

$$R[x]/A=\frac{(U[x]+A)(V[x]+A)}{A} \subseteq A.$$ 

Thus $(U[x]+A)(V[x]+A)\subseteq A$. But since $R[x]/A$ is prime, we have either $U[x]+A=\bar{0}$ or $V[x]+A=\bar{1}$ in $R[x]/A$. Therefore $U \subseteq A$ or $V \subseteq A$. Hence $U\subseteq A \cap R$ or $V \subseteq A \cap R$. This means $\bar{U}=\bar{0}$ or $\bar{V}=\bar{0}$. So $R+\text{ Ann }_R(M)=R/R \cap A$ is a prime ring.

On the other hand, since $R$ is a von Neumann regular ring, its homomorphic image $R/R \cap A$ is also a von Neumann regular ring. Hence the ring $R/R \cap A$ is a prime, von Neumann regular ring satisfying a polynomial identity. So by Lemma 12, $R/R \cap A$ is simple Artinian. Thus the ideal $R \cap A$ is a maximal ideal.
By this argument so far done in the proof we have that \( R/A \) is maximal whenever \( A \) is a primitive ideal of \( R[x] \). But since \( R[x] \) is a P.I.-ring, every primitive ideal of \( R[x] \) is a maximal ideal. So every maximal ideal of \( R[x] \) can be contracted to a maximal ideal of \( R \). But note that in the proof of Proposition 14 every irreducible \( R[x] \)-module is a \text{finite direct sum} of an irreducible \( R \)-module whenever every maximal ideal of \( R \).

Returning to our situation, \( M \) is a finite direct sum of irreducible \( R \)-module. Now finally by condition (ii) since irreducible \( R \)-module is injective, we have that \( M \) is an injective \( R \)-module. This completes the proof.

Example 16. Without P.I.-ness of \( R \), Theorem 15 is not true. Let \( V \) be an infinite dimensional vector space over a field \( K \). Let \( S \) be the socle of \( \text{End}_K(V) \), and let \( R := S + KI \) Then \( R \) is a von Neumann ring but not P.I. In this case as Villamayor and Michler pointed out, as a right \( R \)-module, \( V \) is irreducible but not injective.

References

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