ON AN IDENTITY RELATED TO
THE HOMOMORPHISM
CLOSURE PROPERT Y

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O. Introduction

Let Hom(H,G) be the set of all homomorphisms of a group H into another group G. For \( \varphi \) and \( \psi \) in Hom(H,G), define a mapping \( \varphi + \psi : H \to G \) by \( (\varphi + \psi)(x) = \varphi(x) + \psi(x) \) for all \( x \) in H. It is shown that \( \varphi + \psi \) is again a homomorphism if G is abelian. In fact, Hom(H,G) is a group under this operation for any group H if and only if G is abelian ([3],[4],[5]). Furthermore, Hom(H,G) is abelian for every group H in that case. In this paper, we will generalize the condition on a universal algebra for Hom(B,A) to be an algebra of the same type as A and B. As in the group case, this condition is simply given by an identity, called the medial law. It is assumed that the readers have basic concepts of universal algebra, such as types, subalgebras, homomorphisms, congruences, direct products. For terminology and definitions one can refer to [2] or [7].

1. Preliminary Definitions and Theorems.

A universal algebra (or algebra in short) is a pair \((A, \Omega)\) where \( A \) is a set, called the underlying set of the algebra, and \( \Omega \) is a set of mappings \( f : A^n \to A, \) \( n \) depending on \( f \). Each of these mappings is called an operation of the algebra. If \( f \in \Omega \) is a mapping of \( A^n \) into \( A \), \( n \) is called the arity of \( f \) and denoted by \( a(f) \).

From now on, we will use only the underlying set \( A \) for the algebra

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(A, Ω) dropping out the operations if doing so does not make and confusion.

Let (A, Ω) and (B, Ω) be algebras of the same type and let Hom (B, A) be the set of all homomorphisms of B into A. For every operation \( f \in Ω \) with arity \( n \) and for homomorphisms \( φ_0, φ_2, \ldots, φ_n \) in Hom (B, A) define a mapping \( f (φ_0, φ_2, \ldots, φ_n) : B → A \) by

\[
f(φ_0, φ_2, \ldots, φ_n)(b) = f(φ_0(b), φ_2(b), \ldots, φ_n(b))
\]

for all \( b ∈ B \). If \( f (φ_0, φ_2, \ldots, φ_n) \) is again a homomorphism for each operation \( f \), for all \( φ_0, \ldots, φ_n \) in Hom (B, A) and for every algebra B, we say the algebra A has the homomorphism closure property (HCP for convenience). Thus, in this case, Hom (B, A) is also an algebra of the same type as \( A \) and \( B \).

Let us investigate this property a little farther with groups. If \( H \) and \( G \) are groups, for \( φ \) and \( ψ \) in Hom (H, G), define \( φ + ψ : H → G \) by \( (φ + ψ)(x) = φ(x) \) for all \( x ∈ H \). It can be easily seen that \( φ + ψ \) is also a homomorphism provided G is abelian. That is, every abelian group has the HCP. Under this operation, the zero mapping sending every element of \( H \) onto the identity element of \( G \) is the identity element of Hom (H, G). For \( φ \) in Hom (H, G), define \( -φ(x) = -φ(x) \), and \( -φ \) becomes the inverse of \( φ \) under this operation. The associativity follows trivially. Obviously, \( φ + ψ = ψ + φ \). Thus if \( G \) is abelian, Hom (H, G) is also an abelian group. Conversely, suppose \( G \) has the HCP. Let \( π_1, π_2 : G × G → G \) be the projections given by \( π_1(x, y) = x \) and \( π_2 (x, y) = y \) for all \( x, y ∈ G \). By the HCP of \( G \), \( π_1 + π_2 \) is a homomorphism. Thus for any \( x, y ∈ G \),

\[
x + y = π_1(x, y) + π_2(x, y)
= (π_1 + π_2)(x, y)
= (π_1 + π_2)((o, y) + (x, o))
= (π_1 + π_2)(y, o) + (π_1 + π_2)(o, x)
= π_1(o, y) + π_2(o, y) + π_1(x, o) + π_2(x, o)
= y + x,
\]

which says \( G \) is abelian. Thus, for groups, the HCP is equivalent to being abelian.
2. The HCP and the medial law.

For an $n$-ary operation $f$ and an $m$-ary operation $g$ of an algebra $A$, we say $f$ and $g$ commute each other if

$$f(g(a_{12} \cdots a_{nm}), g(a_{21} a_{22} \cdots a_{2m}), \cdots, g(a_{m1}, a_{m2}, \cdots, a_{m1})) = g(f(a_{11}, a_{21}, \cdots, a_{m1}), f(a_{12}, a_{22}, \cdots, a_{m2}), \cdots, f(a_{1m}, a_{2m}, \cdots, a_{mm}))$$

for all $a_{ij} \in A$, $i=1,2,\cdots, n; j=1,2,\cdots, m$. An algebra is called medial if every pair of operations (not necessarily distinct) commute each other.

A groupoid is a set closed under a binary operation defined on it. For a groupoid, the medial law can be written as $(x+y)+(x+w) = (x+z)+(y+w)$.

**Lemma 2.1.** (Murdoch [9]). Every medial groupoid with an identity element is a commutative monoid.

**Proof** If $o$ is the identity element,

$$x+y = (o+x)+(y+o) = (o+y)+(x+o) = y+x,$$

and the groupoids are commutative, and

$$x+(y+z) = (x+o)+(y+z) = (x+y)+(o+z) = (x+y)+z,$$

which makes the groupoid associative. □

Suppose $A$ is a medial groupoid and $B$ any groupoid. For any $\varphi, \psi \in \text{Hom}(B,A)$ and $x,y \in B$,

$$(\varphi+\psi)(x+y) = \varphi(x+y) + (x+y)$$

$$= (\varphi(x) + \varphi(y)) + (\varphi(x) + \varphi(y))$$

$$= (\varphi(x) + \psi(x) + (\varphi(y) + \psi(y))$$

$$= (\varphi+\psi)(x+y)+ \psi(y).$$

Thus, $\varphi+\psi$ is also a homomorphism and so $A$ has the HCP. In fact, for groupoids, the medial law, not the commutative law, implies the HCP. We state a theorem for groupoids.

**Theorem 2.2.** (Evans [4]). The following four conditions are equivalent
any groupoid $A$.
  i) $A$ is medial;
  ii) $A$ has HCP;
  iii) If $\varphi$ and $\psi$ are in $\text{Hom}(A^2, A)$, then so is $\varphi + \psi$;
  iv) The mapping $(x, y) \mapsto x + y$ is a homomorphism of $A^2$ into $A$.

Generalizing this theorem to general algebraic systems, we have the following theorem.

**Theorem 2.3.** The following four conditions are equivalent for any algebra $A$.

i) $A$ is medial;
ii) $A$ has HCP;
iii) If $f$ is an $n$-ary operation and $\varphi_1, \varphi_2, \ldots, \varphi_n$ are homomorphisms of $A^n$ into $A$, then $f(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is also a homomorphism of $A^n$ into $A$;
iv) If $f$ is an $n$-ary operation on $A$, then $f \in \text{Hom}(A^n, A)$.

**Proof.** i) $\Rightarrow$ ii). Let $f$ be an $n$-ary operation, $B$ be any similar algebra and $\varphi_1, \varphi_2, \ldots, \varphi_n \in \text{Hom}(B, A)$. For any $m$-ary operation $g$ and $b_1 \in B, i = 1, 2, \ldots, m$,

$$f(\varphi_1, \varphi_2, \ldots, \varphi_n)(g(b_1, b_2, \ldots, b_m))$$

$$= f(g(\varphi_1(b_1), \varphi_2(b_2), \ldots, \varphi_n(b_m)))$$

$$= g(f(\varphi_1(b_1), \varphi_2(b_2), \ldots, \varphi_n(b_m)))$$

proving $f(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a homomorphism. ii) $\Rightarrow$ iii). Obvious, iii) $\Rightarrow$ iv). Let $f$ be an $n$-ary operation and $\pi_i$ be the $i$-th projection of $A^n$ onto $A$, then clearly $\pi_i$ is a homomorphism. Then $f(\pi_1, \pi_2, \ldots, \pi_n)$ is a homomorphism of $A^n$ into $A$ and

$$f(\pi_1, \pi_2, \ldots, \pi_n)(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in A$. Thus $(x_1, x_2, \ldots, x_n) \mapsto f(x_1, x_2, \ldots, x_n)$ is a homomorphism of $A^n$ into $A$. iv) $\Rightarrow$ i). The condition iv) simply says that every operation is a homomorphism of a direct product of $A$ into $A$. But, a homomorphism is nothing more than a mapping commuting...
with all operations. Thus, $A$ is medial. □

As we noted, homomorphisms of a group into an abelian group form an abelian group.

Likewise, we say little more for medial algebras.

**Theorem 2.4.** If $A$ is a medial algebra and $B$ is any algebra of the same type, then the set $\text{Hom}(B, A)$ is also a medial algebra of the same type.

**Proof.** $\text{Hom}(B, A)$ is closed under all operations by the preceding theorem. Suppose $f$ and $g$ be $n$-ary and $m$-ary operations, respectively. Let $\varphi_i \in \text{Hom}(B, A)$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then,

$$
(f(g(\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_m}), \ldots, g(\varphi_{j_1}\varphi_{j_2}\cdots\varphi_{j_m}))(b) = f(g(\varphi_{i_1}(b), \varphi_{i_2}(b), \ldots, \varphi_{i_m}(b)), \ldots, g(\varphi_{j_1}(b), \varphi_{j_2}(b), \ldots, \varphi_{j_m}(b)))
$$

$$
=g(f(g(\varphi_{i_1}(b), \varphi_{i_2}(b), \ldots, \varphi_{i_m}(b)), \ldots, f(g(\varphi_{i_1}(b), \varphi_{i_2}(b), \ldots, \varphi_{i_m}(b))))
$$

for all $b \in B$. Thus,

$$
f(g(\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_m}), \ldots, g(\varphi_{j_1}\varphi_{j_2}\cdots\varphi_{j_m})) = g(f(\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_m}), \ldots, f(\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_m}))
$$

and so $f$ and $g$ commute as operations on $\text{Hom}(B, A)$. That is, $\text{Hom}(B, A)$ is medial. □

**Example 2.5.** A vector space $V$ over a field $F$ can be treated as a universal algebra with a binary operation and a unary operation for each element in $F$. The commutativity of the field, the commutativity of the addition of the vector space, and the distributivity make these operations commute one another. That is, vector spaces are medial. Thus, the family of all homomorphisms, which are linear transformations, of a vector space into another is again a vector space.

An algebra $A$ is said to have the endomorphism closure property (ECP, in short) if $f(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is an endomorphism for every operation $f$ and endomorphisms $\varphi_1, \varphi_2, \ldots, \varphi_n$ of $A$. 

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Corollary 2.6. Every medial algebra has the ECP and the family of all endomorphisms of the algebra is also a medial algebra of the same type.

3. The ECP and varieties.

A word (or term) in symbols \( X = \{x_1, x_2, x_3, \ldots \} \) is an expression built up from \( X \) using operations inductively as follow: every element in \( X \) is a word, and if \( f \) is an operation and \( u_k, u_{k+1}, \ldots, u_n \) are words then \( f(u_1, u_2, \ldots, u_n) \) is also a word. It is easy to see that every word \( u \) defines a derived mapping of a cartesian product of \( A \) into \( A \) in the natural way of substitutions. A mapping of an algebra derived from a word is called a term function. In this reason, a word is sometimes called a polynomial. An identity (or law) is an equation \( u = v \) for some words \( u \) and \( v \). An algebra \( (A, \Omega) \) is said to satisfy the identity \( u = v \) if \( u \) and \( v \) derive the same term function, that is, \( u(a_1, a_2, \ldots, a_n) = v(a_1, a_2, \ldots, a_n) \) for all \( a_1, a_2, \ldots, a_n \in A \). We say an algebra satisfies a set of identities if it satisfies every identity in the set. A variety \( V \) is a class of similar algebras such that

i) if \( A \in V \) and \( B \) is a subalgebra of \( A \), then \( B \in V \);

ii) if \( A \in V \) and \( B \) is a homomorphic image of \( A \), then \( B \in V \);

iii) if \( A_i \in V \) for \( i \in I \), then \( \times \{ A_i \mid i \in I \} \in V \).

An algebra \( A \) in a variety \( V \) is called \( V \)-free on a generating set \( X \) provided that every mapping of \( X \) into any algebra \( B \) in \( V \) can be extended to a homomorphism of \( A \). This generating set is called a free generating set of \( A \).

Lemma 3.1 ([1]). Every variety \( V \) has a \( V \)-free algebras on every set of generators, and \( V \)-free algebras are uniquely determined up to isomorphism by the cardinalities of their free generating sets.

Lemma 3.2 ([1]). Let \( V \) be a variety and \( A \) be a \( V \)-free algebra. If \( u(a_1, a_2, \ldots, a_n) = v(a_1, a_2, \ldots, a_n) \) is a relation among elements \( a_1, a_2, \ldots, a_n \) of a free generating set of \( A \).

Then, \( u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n) \) is an identity satisfied by all algebras in \( V \).

By corollary 2.6, every medial algebra has the ECP. However, the ECP does not implies the medial law in general ([3]). Thus,
an algebra may have the ECP while it satisfies 'weaker' identities than the medial law. But for varieties, if all algebras in a variety have the ECP, then every algebra in the variety must be medial, as the following theorem asserts.

**Theorem 3.3.** Let $V$ be a variety of algebras. Then every algebra in $V$ has the ECP if and only if every algebra in $V$ is medial.

**Proof.** The necessity is clear by corollary 2.6. For the converse, suppose every algebra in $V$ has the ECP and let $f$ be an $n$-ary and $g$ be an $m$-ary operation. Take the $V$-free algebra $F$ on the free generating set $\{a_i | i=1,2,\ldots,n,j=1,2,\ldots,m\}$ for each $i=1,2,\ldots,n$, let $\varphi_i$ be the endomorphism of $F$ such that $\varphi_i(a_j) = a_j$ for $j=1, 2,\ldots,m$, which exists due to Lemma 3.1. Then, by the definition of $f(\varphi_1,\varphi_2,\ldots,\varphi_n)$.

$$f(\varphi_1,\varphi_2,\ldots,\varphi_n)(g(a_{i_1},a_{i_2},\ldots,a_{i_m}))$$
$$= f(\varphi_1(g(a_{i_1},a_{i_2},\ldots,a_{i_m})),\ldots,\varphi_n(g(a_{i_1},a_{i_2},\ldots,a_{i_m})))$$
$$= f(g(\varphi_1(a_{i_1}),\varphi_2(a_{i_2}),\ldots,\varphi_n(a_{i_m})),\ldots,g(\varphi_1(a_{i_1}),\varphi_2(a_{i_2}),\ldots,\varphi_n(a_{i_m})))$$
$$= f(g(a_{i_1},a_{i_2},\ldots,a_{i_m}),\ldots,g(a_{i_1},a_{i_2},\ldots,a_{i_m})).$$

On the other hand, since $f(\varphi_1,\varphi_2,\ldots,\varphi_n)$ is an endomorphism,

$$f(\varphi_1,\varphi_2,\ldots,\varphi_n)(g(a_{i_1},a_{i_2},\ldots,a_{i_m}))$$
$$= f(g(\varphi_1(a_{i_1}),\varphi_2(a_{i_2}),\ldots,\varphi_n(a_{i_m})),\ldots,f(\varphi_1(a_{i_1}),\varphi_2(a_{i_2}),\ldots,\varphi_n(a_{i_m})))$$
$$= g(f(a_{i_1},a_{i_2},\ldots,a_{i_m}),\ldots,f(a_{i_1},a_{i_2},\ldots,a_{i_m})).$$

Thus,

$$f(g(a_{i_1},a_{i_2},\ldots,a_{i_m}),\ldots,g(a_{i_1},a_{i_2},\ldots,a_{i_m}))$$
$$= g(f(a_{i_1},a_{i_2},\ldots,a_{i_m}),\ldots,f(a_{i_1},a_{i_2},\ldots,a_{i_m})).$$

By Lemma 3.2, every algebra in $V$ satisfies the identity,

$$f(g(x_{i_1},x_{i_2},\ldots,x_{i_m}),\ldots,g(x_{i_1},x_{i_2},\ldots,x_{i_m}))$$
$$= g(f(x_{i_1},x_{i_2},\ldots,x_{i_m}),\ldots,f(x_{i_1},x_{i_2},\ldots,x_{i_m})).$$

That is, $f$ and $g$ commute each other. Hence every algebra in $V$ is medial. $\square$
Corollary 3.4. The medial law is the weakest law for a variety to have all its members medial.

References


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