ON CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction

Let $A$ denote the class of functions of the form

$$f(z)=1+\sum_{n=2}^\infty a_n z^n$$

(1.1)

which are analytic in the unit disk $U=\{z:|z|<1\}$.

And let $S$ denote the subclass of $A$ consisting of analytic and univalent functions $f(z)$ in the unit disk $U$.

A function $f(z)$ in $S$ is said to be starlike of order $\alpha$ if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (z\in U)$$

(1.2)

for some $\alpha (0<\alpha<1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order $\alpha$. Furthermore, a function $f(z)$ in $S$ is said to be convex of order $\alpha$ if

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (z\in U)$$

(1.3)

for some $\alpha (0<\alpha<1)$. And we denote by $K(\alpha)$ the class of all convex functions of order $\alpha$.

It is well known that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$, and that $S^*(\alpha)\subseteq S^*(\omega)\subseteq S^*$, and $K(\alpha)\subseteq K(\omega)\subseteq K$ for $0\leq\alpha<1$.

These classes $S^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson ([6]), and later were studied by MacGregor([4]) and Dinchuk([5]).

Received Mar. 5, 1989.
Let \((f \ast g)(z)\) be the convolution or Hadamard product of two functions \(f(z)\) and \(g(z)\), that is, if \(f(z)\) is given by (1.1) and \(g(z)\) is given by

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n
\]  

(1.4)

Then

\[
(f \ast g)(z) = \sum_{n=2}^{\infty} a_n b_n z^n
\]  

(1.5)

Let \(T\) denote the subclass of \(S\) consisting of functions \(f(z)\) whose nonzero coefficients, from the second on, are negative. That is, an analytic function \(f(z)\) is in the class \(T\) if it can be expressed as

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0)
\]  

(1.6)

and let \(T^*(\alpha) = T \cap S^*(\alpha)\), \(C(\alpha) := T \cap K(\alpha)\).

The class \(T^*(\alpha)\) and related classes possess some very interesting properties and have been studied by Silverman ([9]) and others. Also, Gupta and Ahmad ([2],[3]) introduced the subclasses of \(T\) and obtained some of the results of Silverman ([9]) for the class \(T^*(\alpha)\) and \(C(\alpha)\).

For a function \(f(z)\) in \(S\), we define

\[
D^0 f(z) = f(z)
\]  

(1.7)

\[
D^1 f(z) = Df(z) = zf'(z), \quad \text{and}
\]  

(1.8)

\[
D^j f(z) = D(D^{j-1} f(z)) \quad (j = 1, 2, \ldots).
\]  

(1.9)

The differential operator \(D^j\) was introduced by Salagean ([7]).

With the help of the differential operator \(D^j\), we say that a function \(f(z)\) belonging to \(T\) is said to be in the class \(T(j,\alpha)\) if and only if

\[
\Re \left\{ \frac{D^{j+\alpha} f(z)}{D^j f(z)} \right\} > \alpha \quad (j = 0, 1, 2, \ldots)
\]  

(1.10)

for some \(\alpha(0 \leq \alpha < 1)\), and for all \(z\) in \(U\).
In particular, the class $T(0,\alpha)$ and $T(1,\alpha)$ was studied by Silverman ([9]).

In the paper, we investigate coefficient estimates and distortion properties for the class $T(j,\alpha)$. Furthermore, we prove that the class $T(j,\alpha)$ is closed under convex linear combinations. Also, we generalize some results of Silverman ([9]) and Schild and Silverman ([8]).

2. Coefficient estimates

Theorem 2.1. Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$. If $\sum_{n=2}^{\infty} n(n-\alpha) a_n < 1-\alpha$, then

$$\Re\left\{ \frac{D^{n+1}f(z)}{D^nf(z)} \right\} \alpha(j=0,1,2,\cdots)$$

Proof. It suffices to show that the values for

$$\frac{D^{n+1}f(z)}{D^nf(z)}$$

lie in a circle centered at $w=1$ whose radius is $1-\alpha$

We have

$$\left| \frac{D^nf(z)}{D^{n+1}f(z)} - 1 \right| = \left| \frac{D^nf(z)}{D^{n+1}f(z) - D^nf(z)} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n z^n}{\left| z + \sum_{n=2}^{\infty} n|a_n| z^n \right|^n}$$

$$\leq \frac{\sum_{n=2}^{\infty} n(n-1)|a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n|a_n| |z|^{n-1}}$$

$$\leq \frac{\sum_{n=2}^{\infty} n(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}$$

This last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq (1-\alpha)(1 - \sum_{n=2}^{\infty} n|a_n|)$$
which is equivalent to
\[
\sum_{n=2}^{\infty} n^{l(n-\alpha)}|a_n| \leq 1-\alpha
\]  \hspace{1cm} (2.1)

But (2.1) is true by hypothesis. Hence \(\left| \frac{D^n f(z)}{Df(z)} \right| - 1 \leq 1-\alpha\), and the theorem is proved.

The following theorem give a necessary and sufficient condition for a function to be in \(T(j, \alpha)\).

**Theorem 2.2.** A function \(f(z) = z - \sum_{n=2}^{\infty} a_n z^n\) is in \(T(j, \alpha)\) if and only if
\[
\sum_{n=2}^{\infty} n^{l(n-\alpha)}a_n \leq 1-\alpha.
\]  \hspace{1cm} (2.2)

**Proof** In view of theorem 2.1, it suffices to show the only if part. Assume that
\[
\text{Re}\left\{ \frac{D^n f(z)}{Df(z)} \right\} = \text{Re} \left\{ \frac{z - \sum_{n=2}^{\infty} n^{l(n-\alpha)}a_n z^n}{z - \sum_{n=2}^{\infty} n^l a_n z^n} \right\} > \alpha(|z| < 1). \tag{2.3}
\]

Choose values of \(z\) on the real axis so that \(\frac{D^n f(z)}{Df(z)}\) is real. Upon clearing the denominator in (2.3) and letting \(z \to 1\) through real values, we obtain
\[
1 - \sum_{n=2}^{\infty} n^l a_n \geq \alpha(1 - \sum_{n=2}^{\infty} n^l a_n).
\]

Thus \(\sum_{n=2}^{\infty} n^{l(n-\alpha)}a_n \leq 1-\alpha\), and the proof is complete. Further, the equality in (2.3) is attained for the function \(f(z)\) given by
\[
f(z) = z - \frac{1-\alpha}{n^l(n-\alpha)} z^n(z \geq 2). \tag{2.4}
\]

Using the Theorem 2.2, we have the following.
Corollary 2.3. $T(j+1,a) \subset T(j,a)$ for $j=0, 1, 2, \cdots$ and $0 \leq a \leq 1$.

Corollary 2.4. Let the function $f(z)$ defined by (1.6) be in the class $T(j,a)$, then

$$a_n \leq \frac{1-a}{n!(n-a)} (n \geq 2) \quad (2.5)$$

3. Distortion Theorems

Using the result obtained in Theorem 2.2, we prove

**Theorem 3.1.** If $f(z) \in T(j,a)$, then

$$r - \frac{1-a}{2^j(2-a)} r^2 \leq |f(z)| \leq r + \frac{1-a}{2^j(2-a)} r^2 \quad (|z| = r)$$

with equality for the function $f(z) = z - \frac{1-a}{2^j(2-a)} z^2$.

**Proof.** Note that

$$\sum_{n=2}^{\infty} n!(n-a)a_n \leq 1 - a.$$

This last inequality follows from Theorem 2.2.

Thus

$$|f(z)| \leq r + \sum_{n=1}^{\infty} a_n r^n \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1-a}{2^j(2-a)} r^2.$$

Similarly,

$$|f(z)| \geq r - \sum_{n=1}^{\infty} a_n r^n \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{1-a}{2^j(2-a)} r^2.$$

**Theorem 3.2.** If $f(z) \in T(j,a)$, then

$$r - \frac{1-a}{2^j(2-a)} r^2 \leq |f'(z)| \leq 1 + \frac{1-a}{2^j(2-a)} r$$
with equality for the function \( f(z) = z - \frac{1-\alpha}{2^l(2-\alpha)} z^2 \).

**Proof.** We have

\[
|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n a_n. \tag{3.1}
\]

In view of Theorem 2.2,

\[
2^l \sum_{n=2}^{\infty} n a_n \leq 1 - \alpha + a2^l \sum_{n=2}^{\infty} a_n \\
\leq 1 - \alpha + a2^l \frac{1-\alpha}{2^l(2-\alpha)} = \frac{2(1-\alpha)}{2-\alpha}.
\]

A substitution of (3.2) into (3.1) yields the right-hand inequality.

On the other hand,

\[
|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} n a_n \geq 1 - \frac{1-\alpha}{2^{l-1}(2-\alpha)} r.
\]

This completes the proof.

4. **Convex Linear Combinations**

In this section, we shall prove the class \( T(\gamma, \alpha) \) is closed under convex linear combinations.

**Theorem 4.1.** \( T(\gamma, \alpha) \) is a convex set.

**Proof.** Let the functions

\[
f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n (a_{i,n} \geq 0, \ i=1,2) \tag{4.1}
\]

be in the class \( T(\gamma, \alpha) \). It is sufficient to show that the function \( h(z) \) defined by

\[
h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \ (0 \leq \lambda \leq 1) \tag{4.2}
\]
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is in the class \( T(j, \alpha) \). Since

\[
    h(z) = z - \sum_{n=1}^{\infty} \left( \lambda a_{in} + (1-\lambda) a_{n} \right) z^n,
\]

(4.3)

with the aid of Theorem 2.2, we have

\[
    \sum_{n=2}^{\infty} n^{j(n-\alpha)} \left( \lambda a_{in} + (1-\lambda) a_{n} \right) \leq 1 - \alpha
\]

(4.4)

which implies \( h(z) \in T(j, \alpha) \).

**Theorem 4.2.** Let \( f_1(z) = z \) and

\[
    f_n(z) = z - \frac{1-\alpha}{n! (n-\alpha)} z^n \quad (j=0,1,2,\ldots, \ n \geq 2)
\]

(4.5)

for \( 0 \leq \alpha < 1 \). Then \( f(z) \) is in the class \( T(j, \alpha) \) if and only if it can be expressed in the form

\[
    f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)
\]

(4.6)

where \( \lambda_n \geq 0 \) (\( n \geq 1 \)) and \( \sum_{n=1}^{\infty} \lambda_n = 1 \).

**Proof.** Assume that

\[
    f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) = z - \sum_{n=1}^{\infty} \frac{1-\lambda}{n! (n-\alpha)} \lambda_n z^n.
\]

(4.7)

Then it follows that

\[
    \sum_{n=2}^{\infty} n!(n-\alpha)(1-\alpha) \lambda_n
\]

\[
= \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_2 \leq 1
\]

which shows \( f(z) \in T(j, \alpha) \).

Conversely, assume that the function \( f(z) \) defined by (1.6) belongs to the class \( T(j, \alpha) \).

Then, since

\[
    a_n \leq \frac{1-\alpha}{n! (n-\alpha)} \quad (n \geq 2),
\]

(4.9)
we may put
\[ \lambda_n = n! (n-\alpha) \frac{a_n}{1-\alpha} \quad (n \geq 2), \]  \hfill (4.10)
and
\[ \lambda_i = 1 - \sum_{n=2}^{\infty} \lambda_n. \]  \hfill (4.11)
Hence, we can see that \( f(z) \) can be expressed in the form (4.6).

Above Theorem 4.2 shows that the class \( T(j,\alpha) \) is closed under convex linear combinations.

**Corollary 4.3** The extreme points of the class \( T(j,\alpha) \) are the functions \( f_n(z) \) \((n \geq 1)\) given by Theorem 4.2.

**5. Convolution Properties**

Let the function \( f_i(z) \) \((i=1, 2)\) be defined by (4.1). Then, we define the modified convolution \( (f_1 * f_2)(z) \) of \( f_1(z) \) and \( f_2(z) \) by
\[ (f_1 * f_2)(z) := z - \sum_{n=2}^{\infty} a_{1,n} a_{2,n} z^n \]  \hfill (5.1)

**Theorem 5.1** Let the function \( f_i(z) \) \((i=1, 2)\) defined by (4.1) be in the class \( T(j,\alpha) \). Then \( (f_1 * f_2)(z) \) is in the class \( T(j,\beta) \), where
\[ \beta = \frac{2^j - 2 (\frac{1-\alpha}{2-\alpha})^2}{2^j - (\frac{1-\alpha}{2-\alpha})^2} \]  \hfill (5.2)
The result is sharp.

**Proof.** From Theorem 2.2, we know that
\[ \sum_{n=3}^{\infty} n! (n-\alpha) a_n \leq 1 - \alpha, \text{ and } \sum_{n=3}^{\infty} n! (n-\alpha) b_n \leq 1 - \alpha \]
We wish to find the largest \( \beta = \beta(\alpha) \) such that
\[ \sum_{n=2}^{\infty} n! (n-\beta) a_n b_n \leq 1 - \beta. \]
Equivalently, we want to show that

$$\sum_{n=2}^{\infty} \frac{n^l(n-\alpha)}{1-\alpha} a_n \leq 1$$

(5.3)

and

$$\sum_{n=2}^{\infty} \frac{n^l(n-\alpha)}{1-\alpha} b_n \leq 1$$

(5.4)

imply that

$$\sum_{n=2}^{\infty} \frac{n^l(n-\beta)}{1-\beta} a_n b_n \leq 1$$

(5.5)

for all $\beta=\beta(a)$.

From (5.3) and (5.4), we get, by means of the Cauchy-Schwarz inequality,

$$\sum_{n=2}^{\infty} \frac{n^l(n-\alpha)}{1-\alpha} \sqrt{a_n b_n} \leq 1$$

(5.6)

It will be, therefore, sufficient to prove that

$$\frac{n^l(n-\beta)}{1-\beta} a_n b_n \leq \frac{n^l(n-\alpha)}{1-\alpha} \sqrt{a_n b_n}$$

for $\beta=\beta(a)$ and $n=2, 3, \ldots$

or

$$\sqrt{a_n b_n} \leq (\frac{n-\alpha}{n-\beta}) (\frac{1-\beta}{1-\alpha})$$

From (5.6), it follows that $\sqrt{a_n b_n} \leq \frac{1-\alpha}{n^l(n-\alpha)}$ for each $n$ ($n=2, 3, \ldots$).

Hence, it will be sufficient to show that

$$\frac{1-\alpha}{n^l(n-\alpha)} \leq (\frac{n-\alpha}{1-\alpha}) (\frac{1-\beta}{n-\alpha})$$

for all $n=2, 3, \ldots$. (5.7)
Inequality (5.7) is equivalent to
\[
\beta \leq \frac{n^j - n(\frac{1}{n} - \alpha)^2}{n^j - (\frac{1}{n} - \alpha)^2} = \frac{n^j(n - \alpha)^2 - n(1 - \alpha)^2}{n^j(n - \alpha)^2 - (1 - \alpha)^2}
\] (5.8)

The right hand side of (5.8) is an increasing function of \( n \) (\( n=2,3,\cdots \)). Therefore, setting \( n=2 \) in (5.8), we get
\[
\beta \leq \frac{2^j - 2(\frac{1}{2} - \alpha)^2}{2^j - (\frac{1}{2} - \alpha)^2}
\]

The result is sharp, for the functions
\[
f_i(z) = z - \frac{1-\alpha}{2^i(2-\alpha)} \quad z \in T(j,\alpha) \quad (j=1,2)
\] (5.9)

References

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