ON IDEALS OF ENDMORPHISM RING OF PROJECTIVE MODULE

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0. Introduction

The object of the paper is to study the relationship between submodules of projective module and ideals of endomorphism ring of projective module. In a projective module \( _RM \) if \( _RM \) has a small submodule, then the endomorphism ring \( \text{End}( _RM ) \) has a small left ideal. If \( _RM \) has the largest submodule, then \( \text{End}( _RM ) \) is a local ring.

Throughout this paper, every ring is an associative ring with identity and every module is a left module. For an element \( a \) in a ring \( R \), \( ^1(a) \) means the left ideal generated by \( a \), in fact, \( ^1(a) = Ra + Za \). The ring of \( R \)-endomorphisms of a left \( R \)-module \( _RM \), denoted by \( \text{End}( _RM ) \), will be written on the right side of \( M \) as right operators on \( M \), that is, \( _RM \text{End}( _RM ) \) will be considered in this paper. For mappings \( f : M \rightarrow N \), \( g : N \rightarrow L \), the composition mapping \( f \cdot M \rightarrow L \) will be written by \( fg \) in order, \( \text{Im}(f) \) is denoted by the image of \( f \).

1. Results

For a submodule \( L \) of a module \( _RM \), consider the set \( I^L \) of all endomorphisms whose images are contained in \( L \), then the zero 0 is in \( I^L \), which says that \( I^L \) is not empty. For each \( f, g \in I^L \), \( \text{Im}(f+g) \leq \text{Im}(f) + \text{Im}(g) \leq L + L \leq L \) and for any \( h \in \text{End}( _RM ) \), \( \text{Im}(hg) \leq \text{Im}(g) \leq L \) so we have a left ideal \( I^L \).

Properties 1.

(1) For any left ideal \( I \) of \( \text{End}( _RM ) \) let \( L = \sum_{i=1}^n \text{Im}(f)_i \), then \( I \leq I^L \)

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(2) If $L_1 \leq L_2 \leq M$, then $I^L_1 \leq I^L_2$ in $\text{End}(R^M)$

(3) $I^M = \text{End}(R^M)$ and $P_0 = 0$

(4) For submodules $L_0(a \in A)$ of $R^M$, $I \cap a \cdot L_0 = a \cap I^L a$

and $\sum_{a \in A} I^L a \leq I^L \cap a$

**Definition 2.** A submodule $L$ of an $R$-module $R^M$ is said to be fully invariant if every endomorphism on $R^M$ sends $L$ into $L$.

Not all submodule of a module need not be fully invariant for example, $\theta \oplus \{0, 2\}$ is not fully invariant of $Z_4 \oplus 0$ And $\{0, 2\}$ is a fully invariant submodule $z_4$.

**Proposition 3.** If $L$ is a fully invariant submodule of $R^M$, then $I^L$ is a both sided ideal of $\text{End}(R^M)$

**Proof.** If suffices to prove that $I^L$ is right sided ideal. Let $f \in I^L$ and $g \in \text{End}(R^M)$ be arbitrary given, then $\text{Im}(fg) = (\text{Im} f)g \leq Lg \leq L$ Since $L$ is fully invariant, which tells that $fg \in I^L$.

**Remark 4.** Every left ideal $I$ of $\text{End}(R^M)$ has a fully invariant submodule $\bigcap \{\text{Ker} f | f \in I\}$. Since for each $x \in \bigcap \{\text{Ker} f | f \in I\}$, $xf = 0$ for every $f \in I$, and $xhf = 0$ for all $h \in \text{End}(R^M)$ because $I$ is a left sided ideal. Hence $xh$ is contained in $\bigcap \{\text{Ker} f | f \in I\}$.

Let $\iota$ be the multiplication by $j$ on $Z_4$, then $\iota$ becomes an endomorphism of $Z_4$. For a submodule $\{0, 2\}$ of $Z_4$ which is fully invariant, we obtain a both sided ideal $I^{\{0, 2\}} = \{\iota_0, \iota_2\}$.

For fully invariant submodules $L_0(a \in A)$ of $R^M$, their sum $\sum_{a \in A} L_0 a$ and intersection $\bigcap \{L_0 a | a \in A\}$ are also fully invariant.

It may happen to exist distinct submodules $L', L''$ of $R^M$ such that $I^{L'} = I^{L''}$ (for example, in the set of real numbers as a $Z$-module, the set $\mathbb{Q}$ of rational numbers and the set $\mathbb{Z}$ of integers are such submodules, i.e., $I^Q = I^Z = 0$), then we are going to take $L$ as their intersection $L' \cap L''$.

Generally, if $I^{L_0} = P_0(a \in A)$, then $L$ will be regarded as the intersection $\bigcap \{L_0 a | a \in A\}$. From now on, in $P$, $L$ means the least submodule of $R^M$ which induces a left ideal $P$. 
A left $R$-module $M$ is said to be projective if for any exact sequence and for any homomorphism $f: M \rightarrow N$ there is an $R$-homomorphism $h: M \rightarrow L$ such that diagram commutes.

\[ \begin{array}{ccc}
  M & \xrightarrow{f} & N \\
  \downarrow{h} & & \downarrow{g} \\
  L & \xrightarrow{\text{Diagram}} & N \\
\end{array} \]

A submodule $L$ of a left module $M$ is said to be small (or superfluous) if for every submodule $K \leq M$, $L + K = M$ implies $K = M$.

**Lemma 5.** Every epimorphism of $\text{End}_R(M)$ is left invertible if $rM$ is projective.

**Proof** This is easily followed by the proposition 5, p83 in [1].

**Theorem 6.** If a submodule $M$ is small, then the left ideal $L$ is small in $\text{End}(M)$.

**Proof.** We need only consider all left ideals of $\text{End}(M)$.

Suppose $I$ is a left ideal of $\text{End}(M)$ such that $L \subseteq I = \text{End}(M)$.

Then the identity $I$ of $\text{End}(M)$ can be written as a sum of $f \in I$ $i \in I^*$, that is $I = f + i$. Thus $M = \text{Im}(f + i) \subseteq \text{Im}f \cup \text{Im}i \subseteq L + \text{Im}i$. By hypothesis, $L$ is small which implies $\text{Im}i = M$. Thus $i$ is an epimorphism which is in $I$. By Lemma 5, $i$ is left invertible, whence $I = \text{End}(M)$.

Let $M$ be a left module. Then the radical of $M$, ([2])

\[ \text{Rad}M = \bigcap \{K \leq M | K \text{ is maximal in } M \} = \sum \{ L \leq M | L \text{ is small in } M \} \]

**Theorem 7.** If a projective module $M$ has the largest submodule $L$, then $\text{End}(M)$ is a local ring, and $M$ has a small submodule.

**Proof.** From the fact that $L$ is largest in $M$, every homomorphic image of non-epimorphism is contained in $L$. Let $J$ be any ideal of $\text{End}(M)$ such that $J \subseteq \text{End}(M)$, then for each $f \in J$, $\text{Im}f \subseteq L$ so that $f \in L^*$. Hence $J \leq L^*$. This implies $L^*$ is the largest left ideal of $\text{End}(M)$. By Proposition 4 in [1] on p57, and Corollary on p58, the radical
of $\text{End}(M)$ is $I^L$ which is a both sided ideal, since the largest ideal is a maximal ideal in a ring. Now it remains to show that $M$ has a small submodule. Since $\text{rad}M = L = \text{sum}$ of small submodules of $M$ and since a sum of submodules in an empty set is zero, thus there is at least one small submodule.

**Theorem 8.** In a projective module $M$, if $L$ is a homomorphic image of endomorphism, then the left ideal $I^L$ is principal.

**Proof.** Let $L = \text{Im} f$ for $f$ in $\text{End}(M)$. If $g \in I^L$, then $\text{Im} g \subseteq \text{Im} f = L$. Considering a diagram

\[
\begin{array}{c}
M \\
\downarrow g \\
\text{Im} f \\
\end{array}
\xrightarrow{f} 
\begin{array}{c}
M \\
\downarrow h \\
\text{Im} f \\
\end{array}
\xrightarrow{h}
\]

there is an $R$-homomorphism $h : M \to M$ such that $g = hf$.

This means $I^L = \{f\}$.

**Corollary 9.** In a projective module $M$, if $L$ is a fully invariant submodule which is an image of an endomorphism, then a both sided ideal $I^L$ is principal in $\text{End}(M)$.

**Proof.** By Proposition 3, $I^L$ is a both sided ideal. Hence $I^L = \{f\}$ in $\text{End}(M)$.

**References**

2. F. Anderson and K.R. Fuller, "Rings and Categories of Modules" New York Springer-Verlag(1973)
3. Roger Ware, "Endomorphism ring of projective modules" Trans. of AMS Vol 156, Nr.1 March 1971, 233–255.
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