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COMPOSITION OF FUNCTIONS WITH CLOSED GRAPHS AND ITS FUNCTION SPACES

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1. Introduction

Let X, Y and Z be (topological) spaces and $f: X \to Y$ a function from X into Y. A function $f: X \to Y$ is said to have a closed graph if and only if (simply, iff) its graph $G(f) = \{(x, f(x)) : x \in X\}$ is closed in the product space $X \times Y$. Every continuous funciton may not have a closed graph and the example of the differentiation operator from C'(0,1) to C(0,1) shows that every function with a closed graph need not be continuous. P. E. Long [2] has discussed properties induced by functions with closed graphs on its domain and range spaces. In [3], its authors have found a sufficient and neccessary condition for functions having a closed graph.

Lemma 1.1[3]. A function $f: X \to Y$ has a closed graph iff for each $x \in X$, $f(x) = \cap\{cl f(U_x) : U_x$ is a neighborhood of x$\}$, where cl denotes the closure operator.

Now questions arise as to whether composition of functions with closed graphs has its graph closed. If not so, under what conditions may the composition have a closed graph? These are mentioned in Section 2. Section 3 is related to properties of some spaces from function spaces of such functions with their graphs closed.

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2. Composition of functions with closed graphs

Example 2.1. Let (N,D) and (N,Cof) be spaces of natural numbers with discrete and cofinite topologies, respectivly. Consider identity functions $i : (N,Cof) \to (N,D)$ and $i^* : (N,D) \to (N,Cof)$. Then i and i* have closed graphs, but its composition $i^* o\, i$ does not have a closed graph because of [3, Proposition 4.1], or [2, Theorem 4].

In positive directions we can get conditions for which composition of functions with closed graphs may have a closed graph. For the sake of convenience, from now on, we will denote a neighborhood(s) (simply, nbd(s)) U of x in the space X by U_x. Thus V_y means a nbd of y in the space Y.

Proposition 2.2. Let $f : X \to Y$ be continuous and $g : Y \to Z$ have a closed graph. Then $G(gof)$ is closed.

Proof. Let (x,z) be a point in X x Z such that $z \neq g(f(x))$ where $y = f(x)$. Since $G(g)$ is closed iff there exist W_z and V_y such that $W_z \cap g(V_y) = \phi$ and f is continuous, for any $y = f(x) \in V_y$ there exists U_x such that $f(U_x) \subset V_y$ thus we have $W_z \cap g(f(U_x)) = \phi$

Proposition 2.3. Let $f : X \to Y$ have a closed graph and $g : Y \to Z$ be a closed function with compact point inverses. Then $G(gof)$ is closed

Proof. Suppose $z \neq g(f(x))$ to use Lemma 1.1. Then for each $y \in in$ $g^{-1}(z)$ and $y \neq f(x)$, there exists a nbd U_x such that $y \notin cl\, f(U_x)$ because $G(f)$ is closed. This implies that there is a nbd V_y such that $V_y \cap f(U_x) = \phi$. Repeating this way for each $y \in g^{-1}(z)$, we have an open cover $\nabla = \{V_y : y \in g^{-1}(z)\}$ for $g^{-1}(z)$ and thus its finite subcover $\nabla_0 = \{V_y i : i = 1,2, \dots n\}$, for g is compact point inverse. Setting $V = \cup_{i=1}^{n} V_y, i$ and $U = \cap_{i=1}^{n} U_x i$ where U_xi is such that $V_y i \cap f(U_x i) = \phi$. Since g is closed, there exists a nbd W_z such that $g^{-1}(z) \subset g^{-1}(W_z) \subset V$. Thus $g^{-1}(W_z) \cap f(U) = \phi$ and so $W_z \cap g(f(U)) = \phi$. Hence $z \neq \cap \{cl\, g(f(U_x)) : U_x$ is a nbd of $x\}$.

Proposition 2.4. Let Y be a regular space, $f: X \to Y$ continuous and $g: Y \to Z$ a closed function with closed point inverses. Then $G(gof)$ is closed.

Proof. Let (x, z) be a point in X x Z such that $z \neq g(f(x))$, or $f(x) \notin g^{-1}(z)$, $g^{-1}(z)$ is closed. Since Y is regular, there exist nbds $V_{f(x)}$ of $f(x)$ and V_z of $g^{-1}(z)$, respectively, such that $V_{f(x)} \cap V_z = \phi$. Since f is continuous and g is closed, there exists nbds U_x in X and W_z in Z such that $f(U_x) \subset V_{f(x)}$ and $g^{-1}(W_z) \subset V_z$. Thus $f(U_x) \cap g^{-1}(W_z) = \phi$. So $g(f(U_x)) \cap W_z = \phi$. Thus $G(gof)$ is closed.

3. Their function spaces

Let $C(X,Y)$ be the set of all continuous functions of X into Y and $C(X,Y)$ will be equipped with the compact-open topology. For spaces X and Y we will denote the following by;

1. $G(X,Y) = \{f: X \to Y :$ f is continuous and G(f) is closed.$\}$
2. $K(X,Y) = \{C_y : X \to Y :$ for y in Y where for any x in X, $C_y(x) = y\}$

Proposition 3.1. A space Y is homeomorphic to a dense subspace $K(X,Y)$ of $C(X,Y)$. (Refer to [1])

Proposition 3.2. If Y is T_1 then $K(X,Y) \subset G(X,Y)$.

Proof. It is enough to show that every constant funtion has a closed graph. For each (x,p) such that $C_y(x) \neq p$, there is a nbd V_p such that $y \notin V_p$ since Y is T_1. This implies $C_y(U_x) \cap V_p = \phi$ for any U_x.

Proposition 3.3. $G(X,Y)$ is T_1-space.

Proof. Let f and g in $G(X,Y)$ with $f \neq g$. Then there is x_o in X such that $f(x_o) = g(x_o)$. Since $G(g)$ is closed, there is a nbd U_{x_o} such that $f(x_o) \notin cl g(U_{x_o})$. Thus $(x_o, Y - cl\, g(U_{x_o}))$ is a nbd of f not containing g. Similarly, we have a nbd of g not containing f.

Proposition 3.4. If Y is T_1, then $G(X,Y)$ is dense in $C(X,Y)$.

Proof. From Proposition 3.1 and 3.2, we have $C(X,Y)=cl\ K(X,Y)=cl\ G(X,Y)$.

Proposition 3.5. Let X be Hausdorff. Then X is connected iff $G(X,X)$ is connected.

Proof. Let X be Hausdorff and connected. Then $G(X,X)=C(X,X)$ and $G(X,X)=C(X,X)$ is Hausdorff from [1, p.140, 258]. By Proposition 3.1, we have $\mathrm{cl}\, K(X,X)=G(X,X)=C(X,X)$ is connected. Conversely, let $G(X,X)$ be connected and assume X is not connected. Then there exist disjoint open sets G and H of X such that $X=G\cup H$. For each $x\in X$ and $f\in G(X,X)$, either $f(x)\in G$, or $f(x)\in H$. This implies that $G(X,X)=(x,G)\cup(x,H)$. Since $(x,G)\cap(x,H)=\phi$, $G(X,X)$ is ~~disconnected~~ which is a contradiction

References

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