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ON FILTER-CONVERGENCE STRUCTURES AND SEQUENTIAL CONVERGENCE STRUCTURES

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1. Introduction

In [3], [6], [7], [8], and [9], the authors introduced convergence structures by filters or sequences and investigated many properties of these convergence structures and convergence spaces(or limit spaces) determined by these convergence structures. It is well-known that a convergence structure determines only a Cech closure operator([2]) which is a weakened form of a Kuratowski closure operator. Indeed, a convergence space may not be a topological space.

In this paper, we shall define filter-convergence structures and sequential convergence structures and investigate properties of these convergence structures. The motivation we define and study filter-convergence structures and sequential convergence structures is that spaces determined by these convergence structures are topological spaces. Moreover, Frechet spaces will be determined completely by sequential convergence structures. Hence these convergence structures are more stronger than other convergence structures defined by many authors as above. So, these convergence structures are more useful to characterize topological spaces and Frechet spaces.

In section 2, we define filter-convergence structures. We shall investigate properties of filter-convergence structures and show that there exists a subcollection $FC[X]$ of the set of all filter-convergence structures on X such that $FC[X]$ and the set of all topologies on X are dual-isomorphic.

In last section, we define sequential convergence structures and study properties of sequential convergence structures. We shall show

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that Frechet spaces are determined completely by sequential convergence structures and there exists a subfamily $SC^*[X]$ of the set of all sequential convergence structures on X such that the set of all Frechet topologies on X and $SC^*[X]$ are dual-isomorphic.

2. Filter-convergence structures

Let X be any non-empty set and let $F[X]$ denote the set of all filters on X. Then, the following theorem is well-known and very useful to characterize topological spaces. This fact will give us a motivation to construct and investigate filter-convergence structures.

Theorem. In a topological space X, the following statements are always true.

(1) If $\mathscr{F}\in F[X]$ and $\{x\}\in\mathscr{F}$ for some $x\in X$, then $\mathscr{F}\to x$ in the space X.

(2) Let $\mathscr{F}_1, \mathscr{F}_2\in F[X]$. If $\mathscr{F}_1\subset\mathscr{F}_2$ and $\mathscr{F}_1\to x$, then $\mathscr{F}_2\to x$.

(3) Let $x\in X$ and $A\subset X$. If $\mathscr{F}\nrightarrow x$ for each $\mathscr{F}\in F[X]$ with $A\in\mathscr{F}$, then $\mathscr{F}'\nrightarrow x$ for each $\mathscr{F}'\in F[X]$ with $\{y\in X \mid \mathscr{F}\to y$ for some $\mathscr{F}\in F[X]$ with $A\in\mathscr{F}\}\subset\mathscr{F}'$.

Definition 2.1. A mapping $q : F[X]\to\mathscr{P}(X)$ is called *a filter-convergence structure on X* if it satisfies the following properties:

(FC 1) If $\mathscr{F}\in F[X]$ and $\{x\}\in\mathscr{F}$, then $x\in q(\mathscr{F})$.

(FC 2) Let $\mathscr{F}_1, \mathscr{F}_2\in F[X]$. If $\mathscr{F}_1\subset\mathscr{F}_2$ and $x\in q(\mathscr{F}_1)$, then $x\in q(\mathscr{F}_2)$.

(FC 3) Let $x\in X$ and $A\subset X$. If $x\notin q(\mathscr{F})$ for each $\mathscr{F}\in F[X]$ with $A\in\mathscr{F}$, then $x\notin q(\mathscr{F}')$ for each $\mathscr{F}'\in F[X]$ with $\{y\in X \mid y\in q(\mathscr{F})$ for some $\mathscr{F}\in F[X]$ with $A\in\mathscr{F}\}\subset\mathscr{F}'$

The following theorem is clear, hence we omit the proof.

Theorem 2.2. Let $(X,\mathscr{T})$ be a topological space. Define a mapping $q_{\mathscr{T}}: F[X]\to\mathscr{P}(X)$ as follows: for each $\mathscr{F}\in F[X]$,

$$q_{\mathscr{T}}(\mathscr{F})=\{x\in X \mid \mathscr{F}\to x \text{ in the space } (X,\mathscr{T})\}.$$

Then, clearly, $q_{\mathscr{F}}$ ia a filter-convergence structure on X.

Theorem 2.3. Let q be a filter-convergence structure on X. Define a mapping $c_q : F[X] \to \mathscr{P}(X)$ as follows : for each subset A of X,

$$c_q(A) = \{x \in X \mid x \in q(\mathscr{F}) \text{ for some } \mathscr{F} \in F[X] \text{ with } A \in \mathscr{F}\}$$

Then c_q is a Kuratowski closure operator on X. That is, (X, c_q) is a topological space.

Proof. It is clear that (1) $c_q(\phi) = \phi$ and (2) for each $A \subset X$, $A \subset c_q(A)$.

(3) Let A be any non-empty subset of X and suppose that $x \notin c_q(A)$. Then $x \notin q(\mathscr{F})$ for each $\mathscr{F} \in F[X]$ with $A \in \mathscr{F}$. By definition of c_q and *(FC 3)*, $x \notin q(\mathscr{F})$ for each $\mathscr{F}' \in F[X]$ with $c_q(A) \in \mathscr{F}'$. Hence we have $x \notin c_q(c_q(A))$ by definition of c_q. By above (2), $c_q(A) \subset c_q(c_q(A))$ is always true. Therefore, $c_q(A) = c_q(c_q(A))$.

(4) Let A and B be non-empty subsets of X. By (2), we have $c_q(A) \cup c_q(B) \subset c_q(A \cup B)$. It is enough to show that $c_q(A \cup B) \subset c_q(A) \cup c_q(B)$. Let $x \in c_q(A \cup B)$. Then, by definition of c_q, $x \in q(\mathscr{F})$ for some $\mathscr{F} \in F[X]$ with $A \cup B \in \mathscr{F}$. Suppose that $F \cap A = \phi$ and $F' \cap B = \phi$ for some elements F and F' of $\mathscr{F}$. Then, since $\mathscr{F}$ is a filter, $F \cap F' \in \mathscr{F}$, and so $F \cap F' \cap A = \phi$ and $F \cap F' \cap B = \phi$. Hence we have $(F \cap F') \cap (A \cup B) = \phi$, which is a contradiction to the fact that $A \cup B \in \mathscr{F}$. Thus, $F \cap A \neq \phi$ or $F \cap B \neq \phi$ for each $F \in \mathscr{F}$. It follows that $\{F \cap A \mid F \in \mathscr{F}\}$ or $\{F \cap B \mid F \in \mathscr{F}\}$ is a filterbase on X. Let $\{F \cap A \mid F \in \mathscr{F}\}$ be a filterbase and let $\mathscr{F}'$ denote the filter on X generated by the filterbase $\{F \cap A \mid F \in \mathscr{F}\}$. Then, clearly, $\mathscr{F} \subset \mathscr{F}'$ and $A \in \mathscr{F}'$. Since $x \in q(\mathscr{F})$, by (FC 2), we have $x \in q(\mathscr{F}')$. Hence $x \in c_q(A)$, and therefore, $x \in c_q(A) \cup c_q(B)$.

Remark. In Theorem 2.3, we don't have any guarantee that for each $x \in X$, $x \in q(\mathscr{N}(x))$, where $\mathscr{N}(x)$ denotes the nbd filter of x in the space (X, c_q). Hence, it need not be true that $x \in q(\mathscr{F})$ iff $\mathscr{F}$ converges to x in (X, c_q) (written $\mathscr{F} \to x$ in (X, c_q)).

Lemma 2.4. In Theorem 2.3, assume that for each $x \in X$, $x \in q(\mathscr{N}(x))$. Then we have, for each $x \in X$,

$$\mathscr{N}(x) = \cap \{\mathscr{F} \in F[X] \mid x \in q(\mathscr{F})\}.$$

Proof. Since $x\in q(\mathcal{N}(x))$, clearly, we have $\mathcal{N}(x)\supset\cap\{\mathcal{F}\in F[X] \mid x\in q(\mathcal{F})\}$.

Conersely, suppose that there exists $N\in\mathcal{N}(x)$ such that $N\notin\cap\{\mathcal{F}\in F[X] \mid x\in q(\mathcal{F})\}$. Then, there is $\mathcal{F}\in F[X]$ such that $x\in q(\mathcal{F})$ and $N\notin\mathcal{F}$. Since $N\notin\mathcal{F}$, int$(N)\notin\mathcal{F}$, where int(N) is the interior of N in (X,c_q). So, F-int$(N)\neq\phi$ for each $F\in\mathcal{F}$. Let $\mathcal{F}'$be the filter on X generated by the filterbase $\{F\text{-}int(N) \mid F\in\mathcal{F}\}$. Then, clearly, we have $\mathcal{F}\subset\mathcal{F}'$and int$(N)\notin\mathcal{F}'$. Let $\mathcal{M}$ be the ultrafilter on X with $\mathcal{F}'\subset\mathcal{M}$. Since int $(N)\notin\mathcal{F}'$and $\mathcal{F}'\subset\mathcal{M}$ and $\mathcal{M}$ is ultra, X-int$(N)\in\mathcal{M}$ and $x\in q(\mathcal{M})$. Hence we have $x\in c_q$(X-int(N)) by definition of c_q, which is a contradiction.

Theorem 2.5. In Theorem 2.3, assume that for each $x\in X$,

$$x\in q(\cap\{\mathcal{F}\in F[X] \mid x\in q(\mathcal{F})\}).$$

Then, $x\in q(\mathcal{F})$ iff $\mathcal{F}\to x$ in (X,c_q).

Proof. If $x\in q(\mathcal{F})$, then $\mathcal{N}(x)\subset\mathcal{F}$ by Lemma 2.4. Hence $\mathcal{F}\to x$ in (X,c_q).

Conversely, if $\mathcal{F}\to x$ in (X,c_q), then $\mathcal{N}(x)\subset\mathcal{F}$ by definition of filter convergence in a topological space. By hypothesis and Lemma 2.4, $x\in q(\mathcal{N}(x))$. Since $\mathcal{N}(x)\subset\mathcal{F}$ and $x\in q(\mathcal{N}(x))$, $x\in q(\mathcal{F})$ by (FC 2).

Let $FC[X]$ be the set of all filter-convergence structures on X satisfying the assumption of Theorem 2.5 and let $T[X]$ be the set of all topologies on X. Define orders $\leq$and $\leq^*$ on $FC[X]$ and $T[X]$ by

$$q_1\leq q_2\leftrightarrow q_1(\mathcal{F})\subset q_2(\mathcal{F}) \text{ for each } \mathcal{F}\in F[X]$$

and

$$\mathcal{T}_1\leq^*\mathcal{T}_2\leftrightarrow\mathcal{T}_1\subset\mathcal{T}_2 \text{ (set inclusion order)},$$

respectively. Then, by Theorem 2.2, Theorem 2.3, and Theorem 2.5, we obtain the following

Corollary 2.6. Two partially ordered sets $(FC[X],\leq)$ and $(T[X],\leq^*)$ are dual-isomorphic under the correspondence $q\to\mathcal{T}_{c_q}$, where

$\mathscr{T}_{c_q} = \{A \subset X \mid c_q(X\text{-}A) = X\text{-}A\}$.

3. Sequential convergence structures

Let X be any non-empty set and let $S[X]$ denote the set of all sequences in X. Sequences in X will be denoted by small Greek letters α,β etc. The k-th term of the sequence α is denoted by $\alpha(k)$. The small Latin letters s, t denote increasing mappings of the natural number set N into itself. The composition $\alpha \circ s$ is the subsequence of α which has $\alpha(s(k))$ as k-th term. Then. The following theorem is also well-known and very useful to characterize first-countable spaces or Frechet spaces. This fact will give us a motivation to define sequential convergence structures.

Definition 3.1. A topological space X is called *Frechet*[1] if the closure of any subset A of X is the set of all limits of sequences in A.

Remark. It is well-known that every first-countable space is Frechet. In [4] and [5], S.P. Franklin investigated properties of Frechet spaces.

Theorem. In a Frechet space X, the following statements are always true.

(1) For each constant sequence (x) in X, (x) converges to x.

(2) If α converges to x, then every subsequence $\alpha \circ s$ of α converges to x.

(3) Let $x \in X$ and $A \subset X$. If for each sequence α in A, α does not converge to x, then for each sequence β in $\{y \in X \mid \alpha$ converges to y, for some α in A$\}$, β also does not converge to x.

Definition 3.2. A non-empty subfamily $\mathscr{L}$ of $S[X] \times X$ is called *a sequential convergence structure on* X if it satisfies the following properties:

(SC 1) For each $x \in X$, $((x), x) \in \mathscr{L}$, where (x) is the constant sequence

whose k-th term is x for all indices k.

(SC 2) If $(\alpha,x)\in\mathscr{L}$, then $(\alpha\circ s,\ x)\in\mathscr{L}$ for each s.

(SC 3) Let $x\in X$ and $A\subset X$. If $(\alpha,x)\notin\mathscr{L}$ for each α in A, then $(\beta,x)\notin\mathscr{L}$ for each β in $\{y\in X \mid (\alpha,y)\in\mathscr{L}$ for some α in $A\}$.

If a sequential convergence structure $\mathscr{L}$ on X is given, then the pair $(X,\mathscr{L})$ is called *a sequential convergence space.*

By definition of Frechet spaces, we obtain the following

Theorem 3.3. Let $(X,\mathscr{T})$ be any Frechet space and let $\mathscr{L}_{\mathscr{T}}$ be the set of all pairs $(\alpha,x)\in S[X]\times X$ such that α converges to x in the space $(X,\mathscr{T})$. Then, $\mathscr{L}_{\mathscr{T}}$ is a sequential convergence structure on X.

Proof. It is straightforward.

Theorem 3.4. Let $\mathscr{L}$ be any sequential convergence structure on X. Define a mapping $c_{\mathscr{L}}: \mathscr{P}(X)\to\mathscr{P}(X)$ as follows: for each subset A of X, $c_{\mathscr{L}}(A)=\{x\in X \mid (\alpha,x)\in\mathscr{L}$ for some α in $A\}$. Then, $c_{\mathscr{L}}$ is a Kuratowski closure operator on X. That is, $(X,c_{\mathscr{L}})$ is a topological space.

Proof. It is clear that (1)$c_{\mathscr{L}}(\phi)=\phi$ and (2)for each subset A of X, $A\subset c_{\mathscr{L}}(A)$.

(3) Let A be any non-empty subset of X and suppose that $x\notin c_{\mathscr{L}}(A)$. Then $(\alpha,x)\notin\mathscr{L}$ for each α in A. By definition of $c_{\mathscr{L}}$ and *(SC 3)*, we have $(\beta,x)\notin\mathscr{L}$ for each β in $c_{\mathscr{L}}(A)$. Hence $x\notin c_{\mathscr{L}}(c_{\mathscr{L}}(A))$. By above (2), $c_{\mathscr{L}}(A)\subset c_{\mathscr{L}}(c_{\mathscr{L}}(A))$ is always true. Therefore, $c_{\mathscr{L}}(A)=c_{\mathscr{L}}(c_{\mathscr{L}}(A))$.

(4) Let A and B be non-empty subsets of X. By (2), we have $c_{\mathscr{L}}(A)\cup c_{\mathscr{L}}(B)\subset c_{\mathscr{L}}(A\cup B)$. It is enough to prove that $c_{\mathscr{L}}(A\cup B)\subset c_{\mathscr{L}}(A)\cup c_{\mathscr{L}}(B)$. Let $x\in c_{\mathscr{L}}(A\cup B)$. The $(\alpha,x)\in\mathscr{L}$ for some α in $A\cup B$. We divide the proof into two cases.

Case 1. The range of α is finite, i.e., $\{\alpha(k) \mid k\in N\}$ is finite. Then there exists an element p of $\{\alpha(k) \mid k\in N\}$ such that $\alpha \circ s=(p)$ for some subsequence $\alpha \circ s$ of α. Since $(\alpha,x)\in \mathscr{L}$, $(\alpha \circ s, x)\in\mathscr{L}$ by *(SC 2)*, and hence $((p), x)\in\mathscr{L}$. From the fact that α is a sequence in $A\cup B$, we have $p\in A$ or $p\in B$. Therefore, by *(SC 1)*, $x\in c_{\mathscr{L}}(A)\cup c_{\mathscr{L}}(B)$.

Case 2. $\{\alpha(k) \mid k\in N\}$ is infinite.
Then $\{\alpha(k) \mid k\in N\}\cap A$ is infinite or $\{\alpha(k) \mid k\in N\}\cap B$ is infinite. If $\{\alpha(k) \mid k\in N\}\cap A$ is infinite, then there exists a subsequence $\alpha \cdot s$ of α in A. Since $(\alpha,x)\in \mathscr{L}$, $(\alpha \circ s, x)\in \mathscr{L}$ by *(SC 2)*, and hence $x\in c_{\mathscr{L}}(A)$. Therefore, $x\in c_{\mathscr{L}}(A) \cup c_{\mathscr{L}}(B)$.

Let $\mathscr{L}$ be a sequential convergence structure on X. Hereafter, we use the notation $\mathscr{L}^*$ for the set of all pairs $(\alpha,x)\in S[X]\times X$ such that α converges to x in the topological space $(X, c_{\mathscr{L}})$. Then, clearly, we have $\mathscr{L}^*$ is also a sequential convergence structure on X, but $\mathscr{L}\neq \mathscr{L}^*$, in general.

Now we investigate relations between $\mathscr{L}$ and $\mathscr{L}^*$.

Lemma 3.5. Let $\mathscr{L}$ be a seuential convergence structure on X. Then, A is a nbd of x in $(X,c_{\mathscr{L}})$ iff for each $(\alpha,x)\in \mathscr{L}$, α is eventually in A.

Proof. Let A be a nbd of x in $(X,c_{\mathscr{L}})$ and $(\alpha,x)\in \mathscr{L}$. Since A is nbd of x in $(X,c_{\mathscr{L}})$, there exists an open set O in $(X,c_{\mathscr{L}})$ such that $x\in O\subset A$. It follows that $c_{\mathscr{L}}(X\text{-}O)=X\text{-}O$, and thus there does not exist β in X-O such that $(\beta,x)\in\mathscr{L}$ by definition of $c_{\mathscr{L}}$. We now prove that $\{k\in N \mid \alpha(k)\in X\text{-}O\}$ is finite. If $\{k\in N \mid \alpha(k)\in X\text{-}O\}$ is infinite, then there exists a subsequence $\alpha \circ s$ of α in $X\text{-}O$. Since $(\alpha,x)\in\mathscr{L}$, $(\alpha \circ s, x)\in\mathscr{L}$ by *(SC 2)*, which is a contradiction. From this fact, we have α is eventually in O. Therefore, α is eventually in A.

Conversely, suppose that there exists a subset A of X such that A is not a nbd of x in $(X,c_{\mathscr{L}})$ and for each $(\alpha,x)\in \mathscr{L}$, α is eventually in A. Then, since α is eventually in A for each $(\alpha,x)\in \mathscr{L}$ and $((x),x)$

$\in \mathscr{L}$ by *(SC 1)*, clearly, we have $x \in A$. Since A is not a nbd of x in $(X, c_{\mathscr{L}})$, $x \in c_{\mathscr{L}}(X\text{-}A)\text{-}(X\text{-}A)$. By definition of $c_{\mathscr{L}}$, there exists a sequence α in $X\text{-}A$ such that $(\alpha, x) \in \mathscr{L}$. Then, by hypothesis, α is eventually in A, which is a contradiction.

Theorem 3.6. Let $\mathscr{L}$ be a sequential convergence structure on X. Then, we have

(1) $\mathscr{L} \subset \mathscr{L}^*$ and

(2) $c_{\mathscr{L}} = c_{\mathscr{L}^*}$.

Proof. (1) Let $(\alpha, x) \in \mathscr{L}$. Then, by Lemma 3.5, for each nbd A of x in the topological space $(X, c_{\mathscr{L}})$, α is eventually in A. Hence α converges to x in $(X, c_{\mathscr{L}})$, and therefore $(\alpha, x) \in \mathscr{L}^*$.

(2) Let A be any non-empty subset of X. Then, by above (1), we have $c_{\mathscr{L}}(A) \subset c_{\mathscr{L}^*}(A)$. Conversely, let $x \in c_{\mathscr{L}^*}(A)$. Then $(\alpha, x) \in \mathscr{L}^*$ for some α in A. By definition of $\mathscr{L}^*$, α converges to x in $(X, c_{\mathscr{L}})$. It follows that $x \in c_{\mathscr{L}}(\{\alpha(k) \mid k \in N\}) \subset c_{\mathscr{L}}(A)$. therefore, $c_{\mathscr{L}} = c_{\mathscr{L}^*}$.

Corollary 3.7. Let $\mathscr{L}_0$ be a sequential convergence structure on X. Then, we have

(1) for each sequential convergence struture $\mathscr{L}$ on X with $c_{\mathscr{L}} = c_{\mathscr{L}_0}$, $\mathscr{L}^* = \mathscr{L}^*_0$. and

(2) $\cup\{\mathscr{L}$ is a sequential convergence structure on X with $c_{\mathscr{L}} = c_{\mathscr{L}_0}\}$ $= \mathscr{L}^*_0$.

Theorem 3.8. Let $\mathscr{L}$ be any sequential convergence structure on X. If $\mathscr{L}$ satisfies the additional two properties : *(SC 4)* and *(SC 5)*, then $\mathscr{L} = \mathscr{L}^*$.

(SC 4) Let $\alpha \in S[X]$. If $(\alpha \circ s, x) \in \mathscr{L}$ for some s such that $N\text{-}\{s(k) \mid k \in N\}$ is finite, then $(\alpha, x) \in \mathscr{L}$.

(SC 5) Let $\alpha \in S[X]$. If there exists a sequence (β_n) in $S[X]$ or a finite subset $\{\beta_1, \beta_2, \cdots \beta_n\}$ of $S[X]$ such that (1) for each i, $\beta_i = \alpha \circ f_i$ for some mapping $f_i : N \to N$(need not be injective or increasing), (2) for each i, $(\beta_i, x) \in \mathscr{L}$, (3) for each $i \neq j$, $f_i(N) \cap f_j(N) = \phi$, and (4)

$\bigcup_{i\in N} f_i(N)=N$(in finite case $\bigcup_{i=1}^{n} f_i(N)=N$), then $(\alpha,x)\in\mathscr{L}$.

Proof. By Theorem 3.6(1), $\mathscr{L}\subset\mathscr{L}^*$.

Conversely, let $(\alpha,x)\in\mathscr{L}^*$: α converges to x in the space $(X,c_{\mathscr{L}})$. We divide the proof into three cases.

Case 1. $\{k\in N \mid ((\alpha(k)),\ x)\in\mathscr{L}\}$ is empty.

Then, since $(\alpha,x)\in\mathscr{L}^*$, $x\in c_{\mathscr{L}}(\{\alpha(k) \mid k\in N\})$ and there exists β_1 in $\{\alpha(k) \mid k\in N\}$ such that $(\beta_1,x)\in\mathscr{L}$. If $\{\beta_1(k) \mid k\in N\}$ is finite, then there exist $k_0\in N$ and a subsequence $\beta_1\circ t$ of β_1 such that $\beta_1\circ t=(\beta_1(k_0))$. Since $(\beta_1,x)\in\mathscr{L}$, $((\beta_1(k_0)),x)\in\mathscr{L}$ by *(SC 2)*, and hence there exists an element $\beta_1(k_0)$ of $\{\alpha(k) \mid k\in N\}$ such that $((\beta_1(k_0)),\ x)\in\mathscr{L}$, which is a contradiction. It follows that $\{\beta_1(k) \mid k\in N\}$ must be infinite. Let $f_1 : N\rightarrow N$ be a mapping (need not be injective or increasing) with $\alpha\circ f_1=\beta_1$. If N-$f_1(N)$ is finite, then, clearly, $(\alpha,x)\in\mathscr{L}$ by *(SC 4)* and finite case of *(SC 5)*. If N-$f_1(N)$ is infinite, then there exists a subsequence $\alpha\circ s_1$ of α such that

$$\alpha(s_1(1))=\alpha(k_1),\ \text{where}\ k_1=\min\{k \mid k\in N\text{-}f_1(N)\}$$
$$\alpha(s_1(2))=\alpha(k_2),\ \text{where}\ k_2=\min\{k \mid k\in(N\text{-}f_1(N))\text{-}\{k_1\}\}$$
$$\vdots$$
$$\alpha(s_1(n))=\alpha(k_n),\ \text{where}\ k_n=\min\{k \mid k\in(N\text{-}f_1(N))\text{-}\{k_1,k_2,\cdots,k_{n-1}\}\}$$
$$\vdots$$

Since $(\alpha,x)\in\mathscr{L}^*$ and $\alpha\circ s_1$ is a subsequence of α, $\alpha\circ s_1$ converges to x in $(X,c_{\mathscr{L}})$, i.e., $(\alpha\circ s_1,x)\in\mathscr{L}^*$ and thus there exists β_2 in $\{\alpha(s_1(k)) \mid k\in N\}$ such that $(\beta_2,x)\in\mathscr{L}$. Let $g_2 : N\rightarrow N$ be a mapping such that $\alpha\circ s_1\circ g_2=\beta_2$, let $s_1\circ g_2=f_2$. If $(N\text{-}f_1(N))\text{-}f_2(N)$ is finite, then $(\alpha\circ s_1,x)\in\mathscr{L}$ by *(SC 4)* and finite case of *(SC 5)*, and therefore, we have $(\alpha,x)\in\mathscr{L}$ by finite case of *(SC 5)*. If $(N\text{-}f_1(N))\text{-}f_2(N)$ is infinite, we can continue with above process. Therefore, $(\alpha,x)\in\mathscr{L}$ by *(SC 4)* and *(SC 5)*.

Case 2. $\{k\in N \mid ((\alpha(k)),\ x)\in\mathscr{L}\}$ is finite $(\neq\phi)$.

Then, let $\{k_1, k_2,\cdots, k_n\}=\{k\in N \mid ((\alpha(k)),x)\in\mathscr{L}\}$ and let $k_0=\max\{k_1,k_2,\cdots, k_n\}$, there exists a subsequence $\alpha\circ s$ of α such that $\alpha(s(k))=\alpha(k_0+k)$

for each $k\in N$. Since $\{k\in N \mid ((\alpha(s(k))),x)\in \mathscr{L}\}=\phi$ by Case 1, we have $(\alpha\circ s, x)\in \mathscr{L}$, and therefore $(\alpha,x)\in \mathscr{L}$ by *(SC 4)*.

Case 3. $\{k\in N \mid ((\alpha(k)),x)\in \mathscr{L}\}$ is infinite.
If N-$\{k\in N \mid ((\alpha(k)),x)\in \mathscr{L}\}$ is finite, then, clearly, we have $(\alpha,x)\in \mathscr{L}$ by *(SC 4)* and *(SC 5)*. If N-$\{k\in N \mid ((\alpha(k)),x)\in \mathscr{L}\}$ is infinite, then there exists a subsequence $\alpha\circ s$ of α such that

$\alpha(s(1))=\alpha(k_1)$, where $k_1=\min\{k \mid k\in N\text{-}\{k\in N \mid ((\alpha(k)),x)\in \mathscr{L}\}\}$

$\alpha(s(2))=\alpha(k_2)$, where $k_2=\min\{k \mid k\in (N\text{-}\{k\in N \mid ((\alpha(k)),x)\in \mathscr{L}\})\text{-}\{k_1\}\}$

$\vdots$

$\alpha(s(n))=\alpha(k_n)$, where $k_n=\min\{k \mid k\in (N\text{-}\{k\in N \mid ((\alpha(k)),x)\in \mathscr{L}\})\text{-}\{k_1,k_2,\cdots, k_{n-1}\}\}$ $\vdots$

Then, clearly, $\{k\in N \mid ((\alpha(s(k))),x)\in \mathscr{L}\}=\phi$, and hence, by Case 1, we have $(\alpha\circ s, x)\in \mathscr{L}$. Therefore, $(\alpha,x)\in \mathscr{L}$ by *(SC 4)* and *(SC 5)*. The proof is complete.

Examle 3.9. Let Q be the rational number set with usual topology. Let $\mathscr{L}_Q$ denote the set of all pairs $(\alpha,x)\in S[Q]\times Q$ such that α converges to x in Q and let $\mathscr{L}=\{((x),x) \mid x\in Q\}\cup\{(\alpha,x)\in S[Q]\times Q \mid \alpha$ converges to x in Q and α is either increasing or decreasing$\}$.
Then we have (1) $\mathscr{L}_Q$ and $\mathscr{L}$ are sequential convergence structures on Q and (2) $\mathscr{L}\subsetneq\mathscr{L}_Q=\mathscr{L}^*=\mathscr{L}^*{}_Q$.

Finally, we shall study relations between sequential convergence structures on X and Frechet topologies on X.

By definitions of $c_{\mathscr{L}}$ and Frechet spaces, the following will be easily verified and hence we omit the proofs.

Theorem 3.10. (1) For each sequential convergence structure $\mathscr{L}$ on X, $(X,c_{\mathscr{L}})$ is a Frechet space.

(2) For each Frechet topology $\mathscr{T}$ on X, $\mathscr{L}_{\mathscr{T}}=\mathscr{L}^*{}_{\mathscr{T}}$ and $\mathscr{L}_{\mathscr{T}}$ is also a sequential convergence structure on X, where $\mathscr{L}_{\mathscr{T}}=\{(\alpha,x)\in S[X]\times X \mid \alpha$ converges to x in $(X,\mathscr{T})\}$.

Corollary 3.11. Let $FT[X]$ be the set of all Frechet topologies on X and let $SC^*[X]=\{\mathscr{L}^* \mid \mathscr{L}$ is a sequential convergence structure on $X\}$. Then, two partially orderes sets $FT[X]$ and $SC^*[X]$ endowed with the set inclusion order are dual-isomorphic under the corresponde-nce $\mathscr{T} \rightarrow \mathscr{L}_{\mathscr{T}}$.

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