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Abian's Order in Near-Rings and Direct Product of Near-Fields

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Abstract

It is shown that a near-ring N which has no nonzero nilpotent elements is a partially ordered set where $x \leq y$ if and only if $yx = x^2$. Also it is shown that $(N, \leq)$ is infinitely distributive for central elements that is $r(\sup x_i) = \sup(rx_i)$ for every central element r of N and any subset $\{x_i\}$ of N. By using some lemmas we showed that a near-ring without nilpotent elements is isomorphic to a direct product of near-fields if and only if N is hyperatomic and orthogonally complete under the condition that every idempotent of N is central.

1. Introduction

In 1970 Alexander Abian introduced an order relation in a semisimple commutative ring. This relation $\leq$is defined by $x \leq y$ iff $xy = x^2$.

By using that relation he showed that a commutative semisimple ring R is isomorphic to a direct product of fields if and only if R is hyperatomic and orthogonally complete. To prove the above theorem is true he showed that relation is infinetly distributive that is $r(\sup x_i) = \sup(rx_i)$ for every subset $\{x_i\}$ of R (1).

M. Charcron extended Abian's results to noncommutative rings.

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He showed that if R has no nilpotent elements in stead of semisimplecity in commutative case, the order relation defined by Abian is partially order. And he also showed that R is isomorphic to a direct product of division rings if and only if *R* is hyperatomic and orthogonally complete whose meanings are same to that of Abian's (4).

H.C.Myung and L.R.Jimenez showed that Abian's results are also true in alternating rings with additional conditions that are $xy=0$ iff $yx=0$ and $xy=xt$ iff $yx=tx$ for every $x,y,t\in R$ (5).

In this paper we studied Abian's order in ner-rings and obtained similar results to other cases studied by above some matematicians.

Recall that $(N,+,\cdot)$ is called a near-ring if the following conditions are satisfied

(a) $(N,+)$ is a group (not necessarily abelian)

(b) $(N,\cdot)$ is a semigroup.

(c) For every $x,y,x\in N$; $(x+y)\cdot z=xz+yz$, that is right distributive law is satisfied.

Since near-ring structure does not satisfy left distributive law n · 0 may be not 0 where 0 is the identity for addition. But throughout this paper, we assume that $n\cdot 0=0$ for $\forall n\varepsilon N$, that is the zera symmeric $No=\{n\varepsilon N \mid n\cdot 0=0\}=N$. Its main examples are the set of functions on an additive group with addition and multiplication defined by following ;

(a) addition+ : $(f+g)(x)=f(x)+g(x)$ for every x in a group.

(b) multiplication : $(f\cdot g)=f(g(x))$ that is multiplication is composition of functions.

Multiplication will in most cases be indicated by juxtaposition : so we write n_1n_2 instead of $n_1\cdot n_2$.

2. Mian Results

Since left distributive law is not satisfied in near-rings we need the following remark.

Remark. If a near-ring N has no non-zero nilpotent elements, the followings are true.

(i) $xy=0$ if and only if $yx=0$ for every $x,y \in N$

(ii) $xy=0$ if and only if $(-x)y=0$

(iii) $xy=0$ implies that $xzy=0$ for every $z \in N$.

(iv) $xyz=0$ implies that $xzyz=0$

Proof. (i) Since $xy=0$ implies $(yx)^2=0$, $yx=0$

(ii) From the fact that $(-x)y=-xy$ for $(-x)y+xy-(-x+x)y=0$.

But $x(-y)$ may be not -xy because the left distibutive law is not satisfied.

(iii) Since $(xzy)^2=0$ for $yx=0$, $xzy=0$.

(iv) is proved by similar method.

Abias's order in near-rings without nonzero nilpotent elements is introduced slightly differently through the following lemma.

Lemma 1. Let an order $\leq$be given by $x \leq y$ if and only if $yx=x^2$ for every $x,y \in$N in a near-ring N. If N has no nonzero nilpotent elements, (N,$\leq$) is a partially ordered set.

Proof. (i) suppose that $x \leq y$ and y$\leq x$. Since $yx=x^2$ and $xy=y^2$ we have $(y-x)x=0$ and $(x-y)y=0$. By remark (i) and (ii) we obtain $x(x-y)=(-y)(x-y)=0$. Thus $0=x(x-y)+(-y)(x-y)=(x-y)^2=0$.
Hence $x-y=0$. From this we know that $\leq$is antisymmetric.

(ii) Suppose that $x \leq y$ and $y \leq z$. Since $(y-z)x=0$ and $(z-y)y=0$, $(z-y)y=(z-y)yx=(z-y)x^2=(z-y)x=0$ for $ab^2=0$ implies $ab=0$.
Thus $zx=yx=x^2$. Hence we obtain $x \leq z$.

By similar claculation we know that $x \leq y$ implies $xz \leq yz$ for every $z \in N$ by remark (iv).

Abian showed that the fact that $x \leq y$ and $u \leq v$ implies that $xu \leq y$ v in his paper(1). But in near-ring case that may be not true. With additional conditions we obtain similar result for near-rings.

Lemma 2. Let N be a near-ring without nonzero nilpotent elements. If $x \in C(N)$ where $C(N)$ is the center of N, then $xu \leq yv$ if $x \leq y$ and $u \leq v$.

Proof. Since $(v\text{-}u)u=0$, $(v\text{-}u)ux=(v\text{-}u)xux=0$. And $(y\text{-}x)x=(y\text{-}x)xv=(y\text{-}x)vx=(y\text{-}x)vux=0$ by assumption. Thus $yvxu=xvxu=vxux=uxux=xuxu=(xu)^2$

If e is a central idempotent in N then we know that $ex\leq x$ for every $x\in N$ for $xex=exe=exex=(ex)^2$. We study the role of idempotents in N with our order relation.

Lemma 3. Let a,s be elements in a near-ing N without nonzero nilpotent elements such that $sa^2=a$. Then the followings are true.

(i) $sas=a$

(ii) $as=sa$ and as is an idempotent.

(iii) If $x\leq$as for some $x\in N$, then x is an idempotent.

Proof. Since $(asa\text{-}a)asa=(asa\text{-}a)a=0$, $asa(asa\text{-}a)\text{-}a(asa\text{-}a)=(aasa\text{-}a)^2=0$. Thus $asa=a$.

(ii) since $asa=a$, $(as)^2=as$. On one hand $(sa\text{-}as)sa=(sa\text{-}as)as=0$ implies $sa(sa\text{-}as)\text{-}as(sa\text{-}as)=(sa\text{-}as)^2=0$.

(iii) Since $asx=x^2$, $(x\text{-}asx)x=0$ for $x^2\text{-}asx^2=asx\text{-}asasx=asx\text{-}asx=0$ by (i). Thus $x(x\text{-}x^2)\text{-}x^2(x\text{-}x^2)=0$ for $(x\text{-}x^2)x=0$.

We define some terminolegies to prove our main theorem.

Definition. A nonzero element a in a nean-ring N is called a hyperatom in N if and only if for every $x\in N$

(i) $x\leq a$ impliex $x=0$ or $x=a$

(ii) $xa\neq 0$ implies $(xa)s=a$ for some element s in N.

We get the following lemma

Lemma 4. Let x be a nonzero element in a neer-ring N without nonzero nilpotent elements. If $a\leq x$ for some nonzero hyperatom a in N, then there exists an idempotent hyperatom e such that $xe\neq 0$.

Proof. Since a is a nonzero hyperatom, $sa^2=a$ for $a^2\neq 0$.

By lemma 3 $sa=as$ and as is an idempotent. We must show that sa is a hyperatom and $sax\neq 0$. At first we assume that $y\leq as$ for some $y \in N$. Then $ya \leq asa = a$ implies $ya = 0$ or $ya = a$ for a is hyperatom. If $ya = 0$ then $0 = yas = yasy = y(asy) = yy^2 = y^3$. Thus $y = 0$. Secondly if $ya \neq a$ then $0 = (y\text{-}as)a = (y\text{-}as)s$. On one hand $(y\text{-}as)y = 0$ for $y \leq as$. Thus $y(y\text{-}as)\text{-}as(y\text{-}as)\text{-}(y\text{-}as)^2 = 0$. This means that $y = as$. Hence the first condition for hyperatom is satisffied. To show that second condition is also satisfied we assume that $yas \neq 0$ for some $y \in N$. Since a is hyperatom there exists an r in N such that $ysar = a$(*in fact* $as = sa$) Thus as is a hyperatom. Finally $xas = a^2s = a \neq 0$. Hence lemma is proved.

Now we study the relation between hyperatom idempotents and near-fields. We get the following lemma.

Leema 5. Let e be a hyperatom idempotent of a near-ring N without nonzero nilpotent elements. Then the followings are true.

(i) Ne is a near-field.

(ii) If b is another hyperatom idempotent of N and b$\in C(N)$, then $eb = be = 0$, that is the set of all central hyperatom ioempotents of N is orthogonal.

Proof. At first we show that Ne is a subgroup under addition. For arbitrary n_1e, $n_2e \in Ne$ $n_1e\text{-}n_2e = (n_1\text{-}n_2)$e by right distributive law. On one hand if $ne \neq O$, then there exists an r in N such that $ner = e$ for e is a hyperatom where e is clearly a right identity of Ne for multiplication. Thus re is right inverse of ne. since there exists a right identity and a right inverse of every nonzero element ne of Ne, Ne is a multiplication group except zero. Thus Ne is a near-field.

(ii) If b is another nonzero central hyperatom idempotent of N, we get be and eb are also idempotents of N.

From these facts $be \leq b$. And $be = 0$ or $be = b$. If $be = b$, by similar method we get $eb = e$ and $e = be = eb = b$. This is contradiction to assumption. Hence $be = eb = 0$.

Definition. Let N be a nearring.

(1) N is called hyperatomic if for every nonzero element r in N there exists a hyperatom a in A such that $a \leq r$

(2) N is called orthogonal complete if supS esists for every orthogonally subset S of N.

We easily know that if N is hyperatomic then for every nonzero element q of N there exists a hyperatom idempotent e such that $qe \neq 0$ by lemma 4. Moreover we know that for every element r of N the $sup\, e_i r = r$ where e_i is central hyperatom idempotents of N. In fact $e_i r \leq r$ for every r in N.

Before we get the main theorem, we prove the following theorem.

Theorem 6. Let $\{x_i\}$ be a subset of N such that supx_i exists. Then for every a in $C(N)$, $sup(x_i a) = sup(x_i)a$ that is infinitely distributive for center of N.

Proof. Let $sup(x_i) = v$. Since $x_i \leq v$ for all i, $x_i a \leq va$.
On one hand, let u be an upper bound of $x_i a$. It is sufficient to show that $va \leq u$. Since $ax_i = x_i a \leq v$, $x_i a \leq va$ and $x_i \leq v$, we get

$(u\text{-}va)x_i a = (u\text{-}v)ax_i = 0$

$((u\text{-}va)a + v)x_i = x_i^2$ for $vx_i = x_i^2$

Thus $x_i \leq (u\text{-}v)a + v$ for all i. By the fact we get $v \leq (u\text{-}va)a + v$.

Hence $((u\text{-}va)a + v)v = v^2$. This means $uav = vaav = avav$ that is $va -$ $av \leq u$. The theorem is proved.

Now we prove the main theorem.

Theorem 7. A near-ring N is isomorphic to a direct product of near-fields if and only if N is hyperatomic and orthogonally complete under the condition that every hyperatom idempotent of N is in the center of N.

Proof. Let f be an isomorphism from N onto a direct product $\prod F_i$ of near-fields F_i. Let r be a nonzero element and let $f(r) = (r_i)_{i \in I}$. Then there exists some $r_j \neq 0$. where $r_j \in F_j$. Let u_j be the unit of F_j. The element a of N given by $a = rf^{-1}((a_i))$ with $a_j = u_j$ and $a_i = 0$ for $i \neq j$ is a hyperatom of R with $a \leq r$.

Suppose that $x \leq a$. If $x \neq 0$ $ax = x^2$. Thus $f((a\text{-}x)x) = 0$. On one hand

$f(a\text{-}x)x) = f(a\text{-}x))f(x) = (f(a)\text{-}f(x))f(x) = (r_i a_i)_{i \in I}\text{-}(x_i)_{i \in I}\}x_i = 0$. But $r_i a_i = 0$ for $i \neq j$ and $r_j u_j = r_j$. We get $x_j = r_j$ and $x_i = 0$ for $i \neq j$. Thus $a = x$.

Secondly let S be an orthogonal subset of N and $f(a) = \{f(s) \mid for\ s \in X\}$. Since S is orthogonal, $f(s)f(t) = 0$ for $s \neq t$. where. $s,t \in S$.
Let $f(s) = (x_i)$ $f(t) = (y_i)$. Then $x_i \neq O$ implies $y_i = 0$ for all i.
Let v be an element of inverse image of k_i where $k_i = x_i$ for some $f(s) = (x_i)$. We know that v is supS for $f(v)f(s) = (f(s))^2$.

Conversely, if R is hyperatomic and orthogonally complete we show that R is isomorphic to the direct product Ne_i where $\{e_i\}$ is the set of all hyperatom idempotents of N. By lemma 5 we know that Ne_i is a near-field for all i. Let f be a maapping f defined by $f(a) = (ae_i)$ from R into $\prod Ne_i$. Then this mapping f is a near-ring homomorphism for $f(a+b) = ((a+b)e_i) = ((ae_i)+(be_i)) = f(a)+f(b)$ and similarly $f(ab) = f(a)f(b)$. We must show that f is one to one and onto. At first if $f(a) = (ae_i) = 0$, then $ae_i = 0$ for all i.
But $sup(ae_i) = sup(e_i a) = a$ implies $a = 0$, Secondly for arbitrary (a_i, e_i) in $\prod Ne_i$, we let a be $sup_i(a_i e_i)$ because the set $\{a_i e_i\}$ is orthogonal. then we get

$$ae_j sup(a_i e_i)e_j = sup(a_i e_i e_j) = a_j e_j$$

Hence $f(a) = (ae_i) = (a_i e_i)$ so that f is onto. The theorem is proved.

To prove the teorem 7 the condition that every hyperatom idempotent of N is central is essential. But if the the theorem 6 is true for every element a in N, then $sup(a_i e_i)e_j = sup(a_i e_i e_j)$ even if e_i is not central. It is question that any other condition instead of centrality satisfies the theorem 6.

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