ZARANTONELLO TYPE INEQUALITIES FOR LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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1. Introduction.

Throughout this paper X denotes a uniformly convex Banach space and C is a bounded closed convex nonempty subset of X. A mapping T: C→X is called a lipschitzian mapping if there exists a positive number k such that

\[ \| Tx - Ty \| \leq k \| x - y \| \quad \text{for every } x, y \in C, \]

and especially nonexpansive in the case of k=1.

Zarantonello's inequality [1] is valid in Hilbert spaces as follows: Let H be a real Hilbert space and C be a bounded closed nonempty convex subset of H. If T is a contractive self-mapping of C, then for all \( x, \in C \) and \( \lambda_i \geq 0, \ i=1,2,\ldots,n \) with \( \sum_{i=1}^{n} \lambda_i = 1 \),

\[ \| T \left( \sum_{i=1}^{n} \lambda_i x_i \right) - \sum_{i=1}^{n} \lambda_i Tx_i \| \leq \sum_{i \leq j \leq n} \lambda_i \lambda_j \left( \| x_i - x_j \| \right) \left( \| Tx_i - Tx_j \| \right) \]


The purpose of this paper is to provide an analogue of Zarantonello's inequality in uniformly convex Banach spaces with lipschitzian mapping.

Received October 23, 1989

* This work is supported by Korean Science and Engineering Foundation in 1988-89.
First we introduce the definition of lipschitzian mappings of strong type($\gamma$) and give a condition of lipschitzian mapping of strong type($\gamma$) and second we prove an analogue of Zarantonello's inequality in Banach spaces with lipschitzian mappings.

2. Lipschitzian mappings of strong type($\gamma$)

We denote by $\Gamma$ the set of strictly increasing convex, hence continuous functions $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0)=0$. In [2], a mapping $T : C \to X$ is said to be strong type($\gamma$) if $\gamma \in \Gamma$ and for all $x,y$ in $C$ and any $c$, $0 \leq c \leq 1$,

$$\gamma(\|cx+(1-c)y - T(x+(1-c)y)\|)
\leq c(1-c)(\|x-y\| - \|Tx-Ty\|).$$

Obviously, every mappings of strong type($\gamma$) is both a mapping of type($\gamma$) and a contraction. But, not every contraction is of strong type($\gamma$).

**Definition.** Let $X$ be a Banach space, $C$ be a bounded closed convex subset of $X$. A mapping $T : C \to X$ is said to be lipschitzian strong type($\gamma$) with lipschitzian constant $k$ if $\gamma \in \Gamma$ and for all $x,y$ in $C$ and any $\lambda$, $0 \leq \lambda \leq 1$,

$$\gamma(\frac{1}{k} \| \lambda x + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|)
\leq \lambda(1-\lambda)(\|x-y\| - \frac{1}{k} \|Tx-Ty\|)$$

(1)

**Theorem 2.1.** If $X$ is a uniformly convex Banach space and $C$ be a bounded closed convex subset of $X$, then there exists a $\gamma \in \Gamma$ such that every lipschitzian mapping $T : C \to X$ is of lipschitzian strong type($\gamma$) with lipschitzian constant $k$. 
Proof. Let \( \delta \) be the modulus of uniform convexity of \( X \):
\[
\delta(t) = \inf \{ 1 - \frac{\| u + v \|}{2} : \| u \| \leq 1, \| v \| \leq 1, \| u - v \| \geq t \}.
\]

Then \( \delta : [0, 2] \to [0, 1] \) is continuous, increasing, \( \delta(0) = 0 \), and \( \delta(t) > 0 \) for \( t > 0 \), while
\[
2 \min(\lambda, 1 - \lambda) \delta(\| u - v \|) \leq 1 - \| \lambda u - (1 - \lambda)v \| \tag{2}
\]
whenever \( 0 \leq \lambda \leq 1 \) and \( \| u \| \leq 1, \| v \| \leq 1 \).

Now we put
\[
d_i(t) = \begin{cases} \frac{1}{2} \int_0^t \delta(s) \, ds & \text{for } 0 \leq t \leq 2 \\ d_i(2) + \delta(2)(t - 2) & \text{for } t > 2, \end{cases}
\]
and
\[
d_i(t) = \frac{1}{2} \int_0^t d_i(s) \, ds,
\]
then it is shown that \( d_i(t) \in \Gamma, d_i(t) \in \Gamma' \), and \( d_i(t) \leq d_i(t) \leq \delta(t) \) for \( 0 \leq t \leq 2 \) and \( d_i(t) \) is two times differentiable and \( \frac{d^2 d_i(t)}{dt^2} \) is increasing for \( t > 0 \). From (2) we have
\[
2\lambda(1 - \lambda) \quad d_i(\| u - v \|) \leq 1 - \| \lambda u - (1 - \lambda)v \| \tag{3}
\]
whenever \( 0 \leq \lambda \leq 1 \) and \( \| u \| \leq 1, \| v \| \leq 1 \). Since it is sufficient to prove (1) for \( 0 < \lambda < 1 \) and \( x \neq y \), taking \( u = \{ T\lambda x + (1 - \lambda)y \} / \| x - y \| \), \( v = \{ T\lambda x + (1 - \lambda)y - Tx \} / \{ (1 - \lambda)k \| x - y \| \} \), we have from (3)
\[
2\lambda(1 - \lambda) \quad d_i(\frac{\| \lambda T\lambda x + (1 - \lambda)y - T\lambda x + (1 - \lambda)y \|}{\lambda(1 - \lambda)k \| x - y \|})
\]
\[ \leq 1 - \frac{\|Tx-Ty\|}{k \|x-y\|}. \]

Hence, let \( M \) denote the diameter of \( C \) and noting that \( \lambda(I-\lambda) k \|x-y\| \leq \frac{kM}{4} \) and \( \frac{d(x)}{\|x\|^2} \) is increasing, we get

\[ \frac{M}{8} \frac{d(x)}{kM} \lambda \|Tx+(1-\lambda)Ty-T(\lambda x+(1-\lambda)y)\| \]

\[ \leq \lambda \|I-\lambda\| \|x-y\| \frac{1}{k} \|Tx-Ty\|. \]

Thus, defining \( \gamma(t) = \frac{M}{8} \frac{d(x)}{M} t \), we get (1). This completes the proof.

3. Zarantonello type inequalities for Lipschitzian mappings

We state here an analogue of Zarantonello's inequality proved by Bruck.

**Lemma (Bruck[4]).** Let \( X \) be a uniformly convex Banach space, \( C \) be a bounded closed convex subset of \( X \). If \( T: C \to X \) is a contraction, then for all \( x, y \in C \), \( \lambda \geq 0 \), \( i = 1, 2, \ldots, n \) with \( \sum_{i=1}^{n} \lambda_i = 1 \) there exists a strictly increasing convex, continuous function \( \gamma: \mathbb{R} \to \mathbb{R} \) which is dependent on \( n \) and \( \gamma(0) = 0 \), such that

\[ \gamma(\|T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i Tx_i\|) \leq \max_{1 \leq i < j \leq n} (\|x_i - x_j\| - \|Tx_i - Tx_j\|). \]

Following Bruck's way, we now show a variant of this lemma for mappings which are not necessarily nonexpansive.

**Theorem 3.1.** Let \( X \) be a uniformly convex Banach space, \( C \) be a bounded closed nonempty convex subset of \( X \). If \( T: C \to X \) is a Lipschitzian mapping with Lipschitzian constant \( k \), then there exists a \( \gamma: \mathbb{R} \to \mathbb{R} \) dependent on \( n \geq 2 \) such that for all \( x, y \in C \) and \( \lambda_i \geq 0 \), \( i = 1, 2, \ldots, n \) with \( \sum_{i=1}^{n} \lambda_i = 1 \),

\[ \gamma\left(\frac{1}{k} \|T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i Tx_i\|\right) \leq \max_{1 \leq i < j \leq n} (\|x_i - x_j\| - \|Tx_i - Tx_j\|) \] (4)
Proof. We shall prove the assertion by induction. We begin by setting $y_2 = y$. Supposing that the assertion is true for any $k$ elements in $C$ and with some $y_2 \in I$, $k<n$, we shall prove (4) by induction on $n$. We put

$$u_j = (1-\lambda_n)x_j + \lambda_n x_n$$
$$u'_j = (1-\lambda_n)Tx_j + \lambda_n Tx_n$$

and $\mu_j = \frac{\lambda_j}{1-\lambda_n}$ for $j=1,2,\ldots,n-1$. Then $u_k \in C$, $\mu_j \geq 0$ for $j=1,2,\ldots,n-1$.

$$\sum_{i=1}^{n-1} \mu_i = 1, \quad \sum_{i=1}^{n} \lambda_i x_i = \sum_{i=1}^{n-1} \mu_i u_i, \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i T x_i = \sum_{i=1}^{n-1} \mu_i u'_i.$$ We lay out the computations as follows:

$$\frac{1}{k} \| T(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i T x_i \| = \frac{1}{k} \| T(\sum_{i=1}^{n-1} \mu_i u_i) - \sum_{i=1}^{n} \mu_i u'_i \|$$

$$\leq \frac{1}{k} \| T(\sum_{i=1}^{n-1} \mu_i u_i) - \sum_{i=1}^{n-1} \mu_i T u_i \|$$

$$+ \frac{1}{k} \sum_{j=1}^{n-1} \mu_j \| Tu_j - u'_j \|.$$ (5)

$$\gamma_n \left( \frac{1}{k} \| T(\sum_{i=1}^{n-1} \mu_i u_i) - \sum_{i=1}^{n-1} \mu_i T u_i \| \right) \leq \max_{1 \leq j \leq x \leq n-1} \| u_j - u_k \|$$

$$\frac{1}{k} \| Tu_j - T u_k \|.$$ (6)

$$\| u_j - u_k \| \leq \frac{1}{k} \| Tu_j - T u_k \| \leq \| u_j - u_k \| - \frac{1}{k} \| Tu_j - T u_k \| + \frac{1}{k} \| u_k - u_k \|$$

$$= \frac{1}{k} \| u_j - u_k \|.$$ (7)

$$\gamma_n \left( \frac{1}{k} \| Tu_j - u'_j \| \right) \leq \| x_j - x_k \| - \frac{1}{k} \| T x_j - T x_k \|$$

$$\| u_j - u_k \| \leq \frac{1}{k} \| u_j - u_k \| \leq \| x_j - x_k \| - \frac{1}{k} \| T x_j - T x_k \|.$$ (8)

$$\| u_j - u_k \| \leq \frac{1}{k} \| u_j - u_k \| \leq \| x_j - x_k \| - \frac{1}{k} \| T x_j - T x_k \|.$$ (9)
Put: \( t = \max \{ \| x_i - x_k \|, -\frac{1}{k} \| T x_i - T x_k \| : 1 \leq i \leq k \leq n \}. \)

Then by (8)

\[
\frac{1}{k} \| T u_i - u_i \| \leq \gamma_2^{-1} (t),
\]

which combined with (9) and used in (7) yields

\[
\| u_i - u_k \| - \frac{1}{k} \| T u_i - T u_k \| \leq \| u_i - u_k \| - \frac{1}{k} \| u_i - u_k \| + \frac{1}{k} \| u_i - T u_i \| + \frac{1}{k} \| u_i - T u_k \| \\
\leq \| x_i - x_k \| - \frac{1}{k} \| T x_i - T x_k \| + \gamma_2^{-1} (t) \\
+ \gamma_2^{-1} (t) \\
\leq t + 2 \gamma_2^{-1} (t). \tag{10}
\]

When used in (6) this yields

\[
\frac{1}{k} \| T (\sum_{j=1}^{n} \mu_i \mathbf{T} T) - \sum_{j=1}^{n} \lambda_j \mathbf{T} \sum_{j=1}^{n} \mu_i \mathbf{T} u_i \| \leq \gamma_2^{-1} (t + 2 \gamma_2^{-1} (t)). \tag{11}
\]

Finally, when (11) is used with (5) we get

\[
\frac{1}{k} \| T (\sum_{j=1}^{n} \lambda_j x_j) - \sum_{j=1}^{n} \lambda_j T x_j \| \leq \frac{1}{k} \| T (\sum_{j=1}^{n} \mu_i \mathbf{T} T) - \sum_{j=1}^{n} \mu_i \mathbf{T} u_i \| \\
+ \frac{1}{k} \sum_{j=1}^{n} \mu_i \| T u_i - u_i \| \\
\leq \gamma_2^{-1} (t + 2 \gamma_2^{-1} (t)) + \frac{1}{k} \gamma_2^{-1} (t).
\]

We define \( \gamma_3^{-1} (t) = \gamma_2^{-1} (t + 2 \gamma_2^{-1} (t)) + \frac{1}{k} \gamma_2^{-1} (t) \). Hence

\[
\gamma_3 \left( \frac{1}{k} \| T (\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i T x_i \| \right) \leq \max \left( \| x_i - x_j \|, -\frac{1}{k} \| T x_i - T x_j \| \right).
\]
We give another proof of the following corollary which was proved by Hirano [3].

**Corollary 3.1.** Let \( C \) be a bounded closed convex subset of \( X \) and \( T : C \to C \) be a nonexpansive mapping. Let \( x \in C \), \( f \in F(T) \), and \( 0 < \alpha \leq \beta < 1 \). Then for each \( \varepsilon > 0 \), there exists \( N > 0 \) such that for all \( n \geq N \),

\[
\| T^k(\lambda T^nx + (1-\lambda)f) - (\lambda T^n*f + (1-\lambda)f) \| < \varepsilon
\]

for all \( k > 0 \) and \( \lambda : \alpha \leq \lambda \leq \beta \).

**Proof.** Since \( T^* : C \to C \) is a nonexpansive mapping, by Theorem 3.1, there exists \( y \in T^*y \) such that

\[
\| \lambda T^*y + (1-\lambda)T^*z - T^*(\lambda y + (1-\lambda)z) \| \\
\leq \gamma(\| y-z \| - \| T^*y - T^*z \|)
\]

for \( y, z \in C \), \( 0 \leq \lambda \leq 1 \). Therefore taking \( y = T^*x \) and \( z = f \), we have

\[
\| \lambda T^*T^*x + (1-\lambda)f - T^*(\lambda T^{*}x + (1-\lambda)f) \| \\
\leq \gamma'(\| T^{*}x - f \| - \| T^{*}y - T^{*}z \|).
\]

Since the sequence \( \{ \| T^*x - f \| \}_{n=0}^{\infty} \) is decreasing, \( \lim_{n \to \infty} \| T^*x - f \| \) exists and for any \( \varepsilon > 0 \) there exists a positive integers \( N \) such that for any integer \( n \geq N \),

\[
\| T^*x - f \| - \| T^{*}x - f \| < \varepsilon.
\]

It follows

\[
\| \lambda T^{*}x + (1-\lambda)f - T^*(\lambda T^{*}x + (1-\lambda)f) \| < \gamma'(\varepsilon).
\]

This completes the proof.
Theorem 3.2. Let $X$ be a Banach space, $C$ be a bounded closed nonempty convex subset of $X$. If $T:C \to X$ is of lipschitzian strong type $(\gamma)$ with lipschitzian constant $k$ for some $\gamma \in \Gamma$, then there exists a $\gamma \in \Gamma$ dependent on $n \geq 2$ such that for all $x \in C$ and $\lambda_i \geq 0$, $i=1, 2, \ldots, n$, with $\sum_{i=1}^{n} \lambda_i = 1$,

$$
\gamma \left( \frac{1}{k} \| T \left( \sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i T x_i \right) \| \right)
\leq \sum_{i_1, i_2 \leq n} \lambda_{i_1} \lambda_{i_2} \left( \| x_{i_1} - x_{i_2} \| - \frac{1}{k} \| T x_{i_1} - T x_{i_2} \| \right) \tag{12}
$$

Proof. We shall prove the assertion by induction. The assumption of $T$ being of lipschitzian strong type $(\gamma)$ shows that (12) is true for $n=2$. We now here note that the $\gamma$ in Theorem 2.1 and the $\gamma_n$ constructed later inductively starting with $\gamma$ are two times differentiable. Thus we may assume for convienence of caculation that any function of $\Gamma$ is two times differentiable. Supposing that the assertion is true for any $k$ elements in $C$ and with some $\gamma \in \Gamma$, $k<n$, we shall prove (12) by induction on $n$. Since there exists at least one $\lambda_i < \frac{1}{2}$ we may assume $0 \leq \lambda_i \leq \frac{1}{2}$. We define $u_i = (1-\lambda_i) x_i + \lambda_i x_n$, $u'_i = (1-\lambda_i) x_i + \lambda_i T x_i$, and $\mu_i = \lambda_i / 1-\lambda_i$ for $j=1, 2, \ldots, n-1$. Then $u_i \in C$, $\mu_i \geq 0$ for $j=1, 2, \ldots, n-1$. Hence $\sum_{j=1}^{n} \mu_i = 1$ and $\sum_{i=1}^{n} \lambda_i x_i = \sum_{j=1}^{n} \mu_i u_i$, $\sum_{i=1}^{n} \lambda_i T x_i = \sum_{j=1}^{n} \mu_i T u_i$. Now we have

$$
\| \frac{1}{k} (T \left( \sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i T x_i \right) \| = \| \frac{1}{k} (T \left( \sum_{j=1}^{n} \mu_i u_i - \sum_{j=1}^{n} \mu_i T u_i \right) \| + \sum_{j=1}^{n} \frac{1}{k} \| T u_i - u'_i \|
= I + II,
$$

where $I = \| \frac{1}{k} (T \left( \sum_{j=1}^{n} \mu_i u_i - \sum_{j=1}^{n} \mu_i T u_i \right) \| \|$ and $II = \sum_{j=1}^{n} \frac{1}{k} \| T u_i - u'_i \|$. 


Puttint $t = \sum_{i=1}^{n} \lambda_i \| x_i - x_n \| - \frac{1}{k} \| T x_n - T x_i \|$), and using the properties of $\gamma$ and $\gamma_{n-1}$ we get

$$\gamma(\Pi) \leq \sum_{i=1}^{n-1} \mu_i \gamma \left( \frac{1}{k} \| T u_i - u_i \| \right)$$

$$\leq \sum_{i=1}^{n-1} \lambda_i \left( \| x_i - x_n \| - \frac{1}{k} \| T x_n - T x_i \| \right)$$

$$\leq t,$$

hence $\Pi \leq \gamma^{-1}(t)$, the inverse function of $\gamma$. On the other hand, by the assumption of induction we obtain

$$\gamma_{n-1}(t) \leq \sum_{i=1}^{n-1} \mu_i \gamma \left( \frac{1}{k} \| T u_i - u_i \| \right)$$

$$\leq \sum_{i=1}^{n-1} \mu_i \left( \| u_i - u_i \| - \frac{1}{k} \| u_i - T u_i \| + \frac{1}{k} \| u_i - T u_i \| \right)$$

$$\leq \sum_{i=1}^{n-1} \frac{\lambda_i}{1 - \lambda_n} \left( \| x_i - x_n \| - \frac{1}{k} \| T x_n - T x_i \| \right) + 2\Pi.$$ 

$$\leq 2t + 2\gamma^{-1}(t),$$

hence $1 \leq \gamma_{n-1}(2t + 2\gamma^{-1}(t))$. Thus

$$\| T \left( \sum_{i=1}^{n} \lambda_i x_i \right) - \sum_{i=1}^{n} \lambda_i T x_i \| \leq \gamma_{n-1}(2t + 2\gamma^{-1}(t)) + \gamma^{-1}(t).$$

We define $\gamma_n^{-1}(t) = \gamma_{n-1}(2t + 2\gamma^{-1}(t)) + \gamma^{-1}(t)$ inductively for $n \geq 3$ with $\gamma_2 = \gamma$. Then as easily checked from the properties of functions $\gamma \in \Gamma$, $k < n$, we have $\gamma_n \in \Gamma$. Hence $\gamma_n^{-1}(2t + \sum_{i=1}^{n} \lambda_i x_i) \leq t$. This completes the proof.

**Corollary 3.2.** Let $X$ be a uniformly convex Banach space, $C$ be a bounded closed nonempty convex subset of $X$. If $T : C \to X$ is a lipschitzian mapping with lipschitzian constant $k$, then for all $x \in C$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \lambda_i = 1$ there exists a $\lambda \in \Gamma$ such
that
\[
\gamma_n \frac{1}{k} \left\| T \left( \sum_{i=1}^{n} \lambda x_i - \sum_{i=1}^{n} \lambda_i T x_i \right) \right\| \leq \sum_{i=1}^{n} \lambda \lambda_i \left( \left\| x_i - x_i \right\| - \frac{1}{k} \left\| T x_i - T x_i \right\| \right).
\]

References


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