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ON DUALITY THEOREMS FOR MULTIOBJECTIVE PROGRAMS

Do Sang Kim and Gue Myung Lee

Abstract

The efficiency(Pareto optimum) is a type of solutions for multiobjective programs. We formulate duality relations for multiobjective nonlinear programs by using the concept of effciency. The results are the weak and strong duality relations for a vector-dual of the Wolfe type involving invex functions.

1. Introduction and Preliminaries

In 1989, Egudo [2] formulated the duality relations for the convex and ρ-convex functions. The purpose of this paper is to establish duality relations between the multiobjective nonlinear program

(MOP) Minimize $f(x)$
subject to $x \in X = \{x \in R^n : g(x) \leq 0\}$;

and the Wolfe vector dual multiobjective program [5]

(WVD) maximize $f(u) + y^t g(u) e$
subject to $(u, \gamma, y) \in Y$, where

$Y = \{(u, \gamma, y) ; \nabla \gamma^t f(u) + \nabla y^t g(u) = 0,\ y \geq 0,\ \gamma > 0$ and $\gamma^t e = 1\}$ and $e = (1, \ldots, 1)^t \in R^p$.

The functions $f: R^n \to R^p$ and $g: R^n \to R^m$ are assumed to be differentiable.

We give the following conventions for vectors in R^n;

$x < y$ if and only if $x_i < y_i$, $i = 1, 2, ..., n$,
$x \leqq y$ if and only if $x_i \leqq y_i$, $i = 1, 2, ..., n$,
$x < y$ if and only if $x \leqq y$ and $x \neq y$,
$x \nleq y$ is the negation of $x \leq y$.

Hanson [3] introduced the following invex function.

Definition 1. Let $h(x) = (h_1(x), ..., h_p(x))^t$; $R^n \to R^p$ be a differentiable function. Then h is invex with respect to η if for all $i = 1, 2, ..., p$, there exists a vector valued function η defined on $R^n \times R^n$ such that for all x, $u \in R^n$,

$$h_i(x) - h_i(u) \geqq \nabla h_i(u) \eta(x, u).$$

We introduce the concept of efficiency(Pareto optimum).

Definition 2. $\bar{x} \in X$ is an efficient solution for *(MOP)* if for all $x \in X$,

$$f(x) \nleq f(\bar{x}).$$

And $(\bar{u}, \bar{\gamma}, \bar{y}) \in Y$ is an efficient solution for *(WVD)* if for all $(u, \gamma, y) \in Y$,

$$f(\bar{u}) + \bar{y}^t g(\bar{u}) e \nleq f(u) + y^t g(u) e.$$

The proof of a strong duality relation will use the following lemma.

Lemma 3 [1]. $\bar{x}$ is an efficient solution for *(MOP)* if and only if for all $k = 1, 2, ..., p$, $\bar{x}$ solves (P_k), where (P_k) ; Minimize $f_k(x)$ subject to $x \in X_k = \{x \in R^n ; f_j(x) \leqq f_j(\bar{x})$ for all $j \neq k$, $g(x) \leqq 0\}$.

2. Duality theorems

Here we establish the weak and strong duality theorems between *(MOP)* and *(WVD)*. First we consider a weak duality relation when the functions are invex.

Theorem 4 (Weak duality). Assume that f and g are invex with respect to η. Then for all $x \in X$ and all $(u,\gamma,y) \in Y$,

$$f(x) \nleq f(u) + y'g(u)e.$$

Proof. Suppose that there exist $x \in X$ and $(u,\gamma,y) \in Y$ such that

$$f(x) \leq f(u) + y'g(u)e.$$

Since $y'g(x)e \leqq 0$, we have

$$f(x) + y'g(x)e \leq f(u) + y'g(u)e$$

This implies

$$\gamma' f(x) + y'g(x) < \gamma' f(u) + y'g(u).$$

Now hypothesis imply $\gamma' f(.) + y'g(.)$ is invex with respect to η. Then we have

$$[\nabla \gamma' f(u) + \nabla y'g(u)]\eta(x,u) < 0.$$

This is a contradiction.

Now we give the Kuhn-Tucker necessary theorem for the singleobjective(i.e.,scalar) program to obtain a strong duality theorem.

Lemma 5[4]. Let $\Theta : R^n \to R$ and g be differentiable functions. Suppose that $\bar{x}$ solves (P) : Minimize $\Theta(x)$ subject to $g(x) \leqq 0$. Assume that $\bar{x}$ satisfies the Slater's constraint qualification (i.e., there exists an $\hat{x} \in X$ such that $g(\hat{x}) < 0$). Then there exists $\bar{y} \in R^m$ such that

$$\nabla\Theta(\bar{x}) + \nabla\bar{y}^t g(\bar{x}) = 0, \quad \bar{y}^t g(\bar{x}) = 0 \text{ and } \bar{y} \geqq 0.$$

Finally we have a strong duality theorem when the functions are invex.

Theorem 6 (Strong duality). Suppose that f and g are invex with respect to η. Let $\bar{x}$ be an efficient solution for (MOP) and assume that $\bar{x}$ satisfies the Slater's constraint qualificatin for (P_k) $k = 1, 2, \ldots, p$. Then there exist $\bar{\gamma} \in R^p$ and $\bar{y} \in R^m$ such that $(\bar{x}, \bar{\gamma}, \bar{y})$ is an efficient solution for (WVD) and $\bar{y}^t g(\bar{x}) = 0$.

Proof. Since $\bar{x}$ is efficient for (MOP), from Lemma 3, $\bar{x}$ solves (P_k) for all $k = 1, 2, \ldots, p$. Now from Lemma 5, there exist $\bar{\gamma} > 0$ and $\bar{y} \geqq 0$ such that

$$\nabla\bar{\gamma}^t f(\bar{x}) + \nabla\bar{y}^t g(\bar{x}) = 0, \quad \bar{y}^t g(\bar{x}) = 0 \text{ and } \bar{\gamma}^t e = 1.$$

Thus $(x, \gamma, y) \in Y$. By the weak duality, for all $(x, \gamma, y) \in Y$,

$$f(\bar{x}) \nleq f(x) + y^t g(x) e.$$

Since $\bar{y}^t g(\bar{x}) = 0$, we have

$$f(\bar{x}) + \bar{y}^t g(\bar{x}) e \nleq f(x) + y^t g(x) e.$$

Hence $(\bar{x}, \bar{\gamma}, \bar{y})$ is an efficient solution for (WVD).

In a subsequent paper, we will study the duality relations between *(MOP)* and the Mond-Weir vector dual program.

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National Fisheries University of Pusan
Pusan 608−737, Korea.
and
Pusan National Institute of Technology
Pusan 608−739, Korea.