ON DUALITY THEOREMS FOR MULTIOBJECTIVE PROGRAMS

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Abstract

The efficiency (Pareto optimum) is a type of solutions for multiobjective programs. We formulate duality relations for multiobjective nonlinear programs by using the concept of efficiency. The results are the weak and strong duality relations for a vector-dual of the Wolfe type involving invex functions.

1. Introduction and Preliminaries

In 1989, Egudo [2] formulated the duality relations for the convex and ρ-convex functions. The purpose of this paper is to establish duality relations between the multiobjective nonlinear program

\( (MOP) \)  
\[ \text{Minimize } f(x) \]  
subject to \( x \in X = \{ x \in \mathbb{R}^n : g(x) \leq 0 \} \);

and the Wolfe vector dual multiobjective program [5]

\( (WVD) \)  
\[ \text{maximize } f(u) + y'g(u)e \]  
subject to \((u, y, y) \in Y\), where

\[ Y = \{(u, y, y) : \nabla y f(u) + \nabla y g(u) = 0, \ y \geq 0, \ y > 0 \text{ and } y e = 1 \} \text{ and } e = (1, \ldots, 1) \]  
\[ \in \mathbb{R}^n. \]
The functions $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are assumed to be differentiable.

We give the following conventions for vectors in $\mathbb{R}^n$:

- $x < y$ if and only if $x_i < y_i$, $i = 1, 2, ..., n$.
- $x \leq y$ if and only if $x_i \leq y_i$, $i = 1, 2, ..., n$.
- $x < y$ if and only if $x \leq y$ and $x \not= y$.
- $x \not< y$ is the negation of $x \leq y$.

Hanson [3] introduced the following invex function.

**Definition 1.** Let $h(x) = (h_1(x), ..., h_p(x))^t : \mathbb{R}^n \rightarrow \mathbb{R}^p$ be a differentiable function. Then $h$ is invex with respect to $\eta$ if for all $i = 1, 2, ..., p$, there exists a vector valued function $\eta$ defined on $\mathbb{R}^n \times \mathbb{R}^p$ such that for all $x$, $u \in \mathbb{R}^n$,

$$h_i(x) - h_i(u) \geq \nabla h_i(u) \eta(x, u).$$

We introduce the concept of efficiency (Pareto optimum).

**Definition 2.** $\bar{x} \in X$ is an efficient solution for (MOP) if for all $x \in X$,

$$f(x) \not< f(\bar{x}).$$

And $(\bar{u}, \bar{y}) \in Y$ is an efficient solution for (WVD) if for all $(u, y) \in Y$,

$$f(\bar{u}) + y^t g(\bar{u}) e \leq f(u) + y^t g(u) e.$$

The proof of a strong duality relation will use the following lemma.

**Lemma 3.** [1] $\bar{x}$ is an efficient solution for (MOP) if and only if for all $k = 1, 2, ..., p$, $\bar{x}$ solves $(P_k)$, where $(P_k)$: Minimize $f_k(x)$ subject to $x \in X_k = \{x \in \mathbb{R}^n : f_j(x) \leq f_j(\bar{x})$ for all $j \neq k, g(x) \leq 0\}$.  


2. Duality theorems

Here we establish the weak and strong duality theorems between (MOP) and (WVD). First we consider a weak duality relation when the functions are invex.

**Theorem 4** (Weak duality). Assume that $f$ and $g$ are invex with respect to \( \eta \). Then for all \( x \in X \) and all \( (u, y) \in Y \),

\[
 f(x) \leq f(u) + y^g(u) e.
\]

**Proof.** Suppose that there exist \( x \in X \) and \( (u, y) \in Y \) such that

\[
 f(x) < f(u) + y^g(u) e.
\]

Since \( y^g(x) e \leq 0 \), we have

\[
 f(x) + y^g(x) e \leq f(u) + y^g(u) e
\]

This implies

\[
 yf(x) + y^g(x) < yf(u) + y^g(u).
\]

Now hypothesis imply \( yf(.) + y^g(.) \) is invex with respect to \( \eta \). Then we have

\[
 [\nabla yf(u) + \nabla y^g(u)] \eta(x, u) < 0.
\]

This is a contradiction.

Now we give the Kuhn-Tucker necessary theorem for the singleobjective (i.e., scalar) program to obtain a strong duality theorem.
Lemma 5[4]. Let $\Theta : \mathbb{R}^n \to \mathbb{R}$ and $g$ be differentiable functions. Suppose that $x$ solves (P): Minimize $\Theta(x)$ subject to $g(x) \leq 0$. Assume that $x$ satisfies the Slater’s constraint qualification (i.e., there exists an $\bar{x} \in X$ such that $g(\bar{x}) < 0$). Then there exists $y \in \mathbb{R}^m$ such that
\[
\nabla \Theta(x) + \nabla y g(x) = 0, \quad y^T g(x) = 0 \quad \text{and} \quad y \geq 0.
\]
Finally we have a strong duality theorem when the functions are invex.

Theorem 6 (Strong duality). Suppose that $f$ and $g$ are invex with respect to $\eta$. Let $x$ be an efficient solution for (MOP) and assume that $x$ satisfies the Slater’s constraint qualification for $(P_k) k = 1, 2, \ldots, p$. Then there exist $y > 0$ and $y^0$ such that $(\bar{x}, y, y^0)$ is an efficient solution for $(WVD)$ and $y^0 g(x) = 0$.

Proof. Since $x$ is efficient for (MOP), from Lemma 3, $x$ solves $(P_k)$ for all $k = 1, 2, \ldots, p$. Now from Lemma 5, there exist $\tilde{y} > 0$ and $y^0 \geq 0$ such that
\[
\nabla \tilde{y} f(x) + \nabla y^0 g(x) = 0, \quad y^0 g(x) = 0 \quad \text{and} \quad y^0 = 1.
\]
Thus $(x, y, y^0) \in Y$. By the weak duality, for all $(x, y, y^0) \in Y$,
\[
f(x) \leq f(x) + y^0 g(x).
\]
Since $y^0 g(x) = 0$, we have
\[
f(x) + y^0 g(x) \leq f(x) + y^0 g(x).
\]
Hence $(x, y, y^0)$ is an efficient solution for (WVD).
In a subsequent paper, we will study the duality relations between (MOP) and the Mond-Weir vector dual program.

References


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