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Star Complex in Inifinite Group*

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1. Introduction

Given a 2-complex, we can get a 1-complex which has variously been called the co-initial graph, star-graph, star complx, and which has proved useful in several contexts [1],[2],[3],[6],[8]. Hyperbolic complexes arise when one considers assigning numbers(weights) to the edges of star-complex of a 2-complex.

In[7], Pride introduced the concept of an involutary presentation. Such presentations are useful when one wants to deal geometrically with groups which have generators of order 2.

In this paper, we will study the problem ; when is 2-cyclically presented group finite or infinite?

The main tool we will make use of is the star complexes of involutary presentations.

By using covering theorem[9], we get involutary presentations from 2-cyclically presented groups. And then we get hyperbolic presentations from those by assigning weights.

2. Definitions and notations

A *1-complex* X consists of two disjoint sets $V=V(X)$(vertices), $E=E(X)$ (edges) and three functions ;

$$\iota : E\to X,\quad \tau : E\to X,\quad ^{-1} : E\to E$$

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satisfying ; $\iota(e)=\tau(e^{-1})$, $(e^{-1})^{-1}=e$ for all $e\in E$

If $e^{-1}=e$ then we will say that e is an *involutary* edge. A *non-empty path* α in X is a sequence $e_1e_2\cdots e_n(n\geq 1)$ of edges with $\tau(e_i)=\iota(e_i+1)(1\leq i<n)$. We define $\iota(\alpha)$, $\tau(\alpha)$ to be $\iota(e_1)$, $\tau(e_n)$. The *length* $L(\alpha)$ of α is n. The *inverse* α^{-1} of α is the path $e_n^{-1}\cdots e_2^{-1}e_1^{-1}$.

The path α is said to be *closed* if $\iota(\alpha)=\tau(\alpha)$. We say that α is *reduced if* $e_i\neq e_{i+1}^{-1}$ for $i=1,\cdots,n-1$.

Moreover it is said to be *cyclically reduced* if all its cyclic permutations are reduced.

A *2-complex* K is an object$\langle X\,;\,\rho_\lambda\,(\lambda\in\Lambda)\rangle$ where X is a 1-complex and ρ_λ are non-empty closed paths in X (called *defining paths)*. We will assume that the defining paths are non-empty and cyclically reduced. A 2-complex with a single vertex is a *presentation*. A presentation may have involutary edges. Such presentation ia called an *involutary presentation*.

We let $R(K)$ denote the set of cyclic permutations of defining paths of K and their inverses.

The *star-complex* K^{st} of a 2-complex K is the 1-complex with vertex set $E(K)$, adge set $R(K)$, and functions

$$\iota^{st}: R(K)\to E(K),\quad \tau^{st}: R(K)\to E(K).$$
$$-1^{st}: R(K)\to R(K)$$

given by

$$\iota^{st}(r)=\text{first edge of } r$$
$$\tau^{st}(r)=\text{ inverse of last edge of } r$$
$$r^{-1st}=r^{-1},\ r\in R(K).$$

A *weight function* on a 1-complex is a mapping θ for the edge set to the real numbers such that $\theta(e)=\theta(e^{-1})$ for each edge e.

If $r=e_1e_2\cdots e_n$ is edge, then we define $\theta(r)=\Sigma_{i=1}^{n}\theta(e_i)$

The interesting situation to us is that when we have a presentation K together with a weight function θ defined on K^{st}.

We denote this situation by (K,θ). We call (K,θ) *hyperbolic* if the following holds.

(1) For any element $e_1 \cdots e_n \in R$,

$$\sum_{i=1}^{n} (1-\theta(e_i e_{i+1} \cdots e_n e_1 \cdots e_{i-1})) > 2.$$

(2) The weight of every non-empty cyclically reduced closed path in K^{st} is at least 2.

(3) There is a non-negative real number N such that every reduced path in K^{st} has weight at least $-N$.

We call (1) and (2) Link condition and Curvature condition. These conditions have clear geometric meanings [2],[6].

We say that a presentation K is *hyperbolic* if (K,θ) is hyperbolic for some θ.

3. Cyclically presented groups

Let $F=\langle x_1,x_2,\cdots,x_n \mid \rangle$ and θ be the automorphism of F induced by permuting the subscripts of the free generators in accordance with thd cycle $(12\cdots n)\in S_n$. For any reduced word $w\in F$, the cyclically presented group $G_n(w)$ is given by

$$G_n(w)=\langle x_1,x_2,\cdots,x_n \mid w, \ w\theta,\cdots,w\theta^{n-1}\rangle$$

Cyclically presented groups comprise a potentially rich source of interesting groups. For example [4],[8], Macdonald groups, Mennicke groups, Fibonacci groups and Higman groups. Since cyclically presented groups have non-negative deficiency, $G_n(w)$ is interesting if and only if it is finite.

To link with one relator products of two cyclic groups, we work only the case $n=2$.

Consider such a group

$$H=\langle x_1,x_2 \ ; R(x_1,x_2)=1, \ R(x_2,x_1)=1\rangle$$

There is an automorphism ϕ of H which interchanges, x_1 and x_2. Thus we can extend H by this automorphism giving the group [5]

$$H^*=\langle x_1,x_2,t\ ;\ t^2=1, t^{-1}x_1t=x_2, R(x_1,x_2)=1, R(x_2,x_1)=1\rangle$$
$$=\langle x_1,t\ ;\ t^2=1,\ \ R(x_1,t^{-1}x_1t)=1\rangle$$

Then $|H^*:H|=2$, so H is infinite if and only if H^* is infinite. Thus to deal with the problem "when is a 2-generator cyclically presented group infinite ? " it suffices to look at groups with presentations of the form(changing notation)

$$\langle a,b \mid a^2=1,\ \ ab^{n_1}ab^{n_2}\cdots ab^{n_r}=1\rangle$$

Let $G=\langle a,b \mid a^2=1, ab^{n_1}ab^{n_2}\cdots ab^{n_r}=1\rangle$, $n=n_1+\cdots n_r$.

Theorem 1. If $r\geq 7$ and $n_1,\cdots,n_r$ are distinct then G is infinite.

Theorem 2. If $n_1=n_2=\cdots=n_r=\alpha$ and $r\geq 5$ then G is infinite.

We define a mapping $\theta : G\to Z_n=\langle t \mid t^n=1\rangle$ by

$$a\to 1$$
$$b\to b$$

then we have an extension homomorphism θ of θ.
Let $Ker\theta=N$ then $|G:N|=n$ and N is generated by $a_0,a_1,\cdots,a_{n-1}$ and b^n, where $a_i=b^iab^i$.
Let $\bar{N}=N/\langle b^n\rangle$ then $\bar{N}$ has the involutary presentation

$$\bar{N}=\langle a_0,a_1,\cdots,a_{n-1} \mid a_i^2=1,\ \ a_i a_{i+n_1}\cdots a_{i+n_1+\cdots+n_r}=1(\text{mod n})\ \ i=0,\cdots,n-1\rangle.$$

So, if $\bar{N}$ is hyperbolic then G is infinite.

Proof of Theorem 1. The star complex of $\bar{N}$ has no closed path of length 2. That is to say, it is a graph. If we assign weight 2/3 to the each edge, then $\bar{N}$ is hyperbolic. Therefore, G is infinite.

Proof of Theorem 2. Each basic path of the involutary star complex of $\bar{N}$ is αr-gon. So, we assign weight $2/r$ to each edge then N is hyperbolic. Therefore, G is infinite.

Theorem 3. If $n_1=\alpha$ and $n_2=\cdots=n_r=1$ then $\frac{1}{a-1}+\frac{1}{r}<\frac{1}{2}$ if and only if G is infinite.

Proof. Since $b^{-(a-1)}=(ba)^r, b^{a-1}$ commutes with a. This is to say, b^{a-1} belongs to the center of G. Therefore, $G/\langle b^{a-1}\rangle$ has the presentation

$$\langle a,b \mid a^t=1, b^{a-1}=1, (ab)^r=1\rangle$$

and is a von Dyck's group $D(2,t-1,r)$. If $|G;\langle b^{a-1}\rangle|$ is finite then $|G;Z(G)|$ is finite. Then by B.H. Neumann's Theorem[4] the derived group G' of G is finite. Since $|G;G'|$ is finite, G is finite. So we have our conclusion.

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