A Pair of Commuting Operators on Hilbert Spaces

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1. Introduction

Since the concept of joint spectrum for a family of operators was initially introduced by R. Arens and A.P. Calderon [1], several authors have established its definitions and properties. The typical and successful definitions among them have carried out by J.L. Taylor [8] and A.T. Dash [6].

In this paper we give a characterization of the joint spectrum, in the sense of J.L. Taylor of a pair of commuting operators on Hilbert spaces and some applications are given.

Let $H$ be a complex Hilbert space and $B(H)$ the algebra of all linear continuous operators on $H$. Let $a=(a_1, a_2) \subset B(H)$ be a pair of commuting operators. Consider the sequence

$$0 \rightarrow H \xrightarrow{\delta^0} H \oplus H \xrightarrow{\delta^1} H \rightarrow 0,$$

where $\delta^0_x (x) = a_1 x + a_2 x$ ($x \in H$) and $\delta^1_x (x_1 \oplus x_2) = a_1 x_1 - a_2 x_2$ ($x_1, x_2 \in H$).

Clearly, $a_1 a_2 = a_2 a_1$ implies $\delta^1 \cdot \delta^0 = 0$. Then, J.L. Taylor has defined $a$ to be nonsingular if the sequence (1.1) is exact; i.e. $\text{im} \, \delta^0 = \ker \delta^1$. And he has defined the joint spectrum $\sigma(a, H)$ of $a$ on $H$, to be the complement of the set of all $z-a = (z_1-a_1, z_2-a_2)$ is nonsingular on $H$.

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2. Invertibility of a commuting pair

We begin the following. Suppose that \(a=(a_1,a_2) \subset B(H)\) is nonsingular on \(H\). Consider the dual sequence of (1.1), namely

\[0 \rightarrow H \xrightarrow{\delta_e^*} H \oplus H \xrightarrow{\delta_2^*} H \rightarrow 0,\]

where \(\delta_e^*(x) = -a_1^* x \oplus a_x^* (x \in H)\) and \(\delta_2^*(x_1 \oplus x_2) = a_1 x_1 + a_2 x_2 (x_1, x_2 \in H)\).

We recall that the pair \(a^*=(a_1^*,a_2^*)\) is nonsingular on \(H\) if the sequence (2.1) is exact.

**Lemma 2.1.** If \(a=(a_1,a_2)\) is nonsingular on \(H\), then both \(a_1^* a_1 + a_2^* a_2\) and \(a_1 a_1^* + a_2 a_2^*\) are invertible on \(H\).

**Proof.** Let us show that \(a_1^* a_1 + a_2^* a_2\) is injective and surjective on \(H\). If \((a_1^* a_1 + a_2^* a_2)x = 0\) for a certain \(x \in H\), then \(a_1 x \oplus a_2 x \in \ker \delta_e^* = (\text{im } \delta_e^*)^\perp\). But \(a_1 x \oplus a_2 x \in \text{im } \delta_e^*\); hence \(a_1 x \oplus a_2 x \in (\text{im } \delta_e^*) \cap \text{im } \delta_e^* = \{0\}\). Thus \(a_1 x = a_2 x = 0\). Since \(\ker \delta_e^* = 0\), we have \(x = 0\). Take an arbitrary \(y \in H\) and let us find an \(x \in H\) such that \(y = a_1^* a_1 x + a_2^* a_2 x\). We infer that \(\delta_e^* : (\ker \delta_e^*)^\perp \rightarrow H\) is an isomorphism, and therefore \(y = \delta_e^* (y_1 \oplus y_2)\) with \(y_1 \oplus y_2 \in (\ker \delta_e^*)^\perp = \text{im } \delta_e^*\); hence \(y_1 \oplus y_2 = a_1 x \oplus a_2 x\). Analogously, the operator \(a_1 a_1^* + a_2 a_2^*\) is invertible and this completes the proof of the lemma.

**Theorem 2.2.** Let \(a=(a_1,a_2) \subset B(H)\) be a commuting pair. Then \(a\) is nonsingular on \(H\) if and only if the operator

\[(2.2) \quad a(a) = \begin{pmatrix} a_1^* & a_2^* \\ -a_2 & a_1 \end{pmatrix}\]

is invertible on \(H \oplus H\).

**Proof.** According to Lemma 2.1, it is clear that the operator
A Pair of Commuting Operators on Hilbert Spaces

(2.3) \[
\begin{pmatrix}
    a_i(a_i^* + a_2^*) & -a_i(a_i^* + a_2) \\
    a_i(a_i^* + a_2) & a_i(a_i + a_2^*)
\end{pmatrix}
\]

is a right inverse for the operator \(\alpha(a)\) given by (2.2): hence \(\alpha(a)\) is surjective on \(H \oplus H\). Let us also notice that \(\alpha(a)\) is injective too. Indeed, if \(\alpha(a)(x_1 \oplus x_2) = 0\), then \(x_1 \oplus x_2 \in \ker \delta_a^0 \cap \im \delta_a^1 = \{0\}\), and hence \(x_1 = x_2 = 0\). Conversely, suppose that \(\alpha(a)\) is invertible on \(H \oplus H\). The \(\alpha(a)^*\) is invertible: therefore

\[
\alpha(a)\alpha(a)^* = \begin{pmatrix}
    a_i^* + a_2^* & 0 \\
    0 & a_i + a_2^*
\end{pmatrix}
\]

is invertible, and hence \((a_i^* + a_2^*)\) and \((a_i + a_2^*)\) are operators from \(B(H)\). Let us prove that the sequence (1.1) is exact. Indeed, if \(\delta^0_0(x) = a_1x \oplus a_2x = 0\), then \((a_i^* + a_2^*)x = 0\), whence \(x = 0\). Assume now that \(\delta^1_0(x_1 \oplus x_2) = a_1x_1 - a_2x_2 = 0\). If \(y = a_i^*x_1 + a_2^*x_2\), then \(\alpha(a)(x_1 \oplus x_2) = y \oplus 0\); hence \(x_1 \oplus x_2 = \alpha(a)^*(y \oplus 0)\), and thus on account of (2.3) we obtain

\[
x_1 = a_i(a_i^* + a_2^*)y, \\
x_2 = a_i(a_i^* + a_2^*)y,
\]

i.e. the exactness of (1.1) at the second step. Finally, if \(y \in H\) is arbitrary, then \(x_i = a_i(a_i^* + a_2^*)y\) \((i=1,2)\) satisfy the equation \(a_1x_1 + a_2x_2 = y\), and the proof is complete.

Notice that \(a = (a_i, a_2) \subset B(H)\) is nonsingular if and only if the matrix

\[
\alpha(a^*) = \begin{pmatrix}
a_i & a_2 \\
-a_2^* & a_1^*
\end{pmatrix}
\]
is invertible on $H \oplus H$, and also if and only if the matrix

$$\alpha(a)^* = \begin{pmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{pmatrix}$$

is invertible on $H \oplus H$.

**Corollary 2.3.** If $A$ is any commutative algebra of operators on $H$, then the map

$$A^2 \ni a \longrightarrow \alpha(a) \in B(H \oplus H)$$

is $R$-linear.

**Proof.** Since the maps $\delta^0$ and $\delta^1$ are linear on $A^2$, $\alpha(a)$ is $R$-linear.

**Remark.** The set of matrices $\{\alpha(z) : z \in C^2\}$ can be identified with the algebra of quaternions and that the map $z \rightarrow \alpha(z)$ is an $R$-linear isometric isomorphism $[10]$.

**Corollary 2.4.** For any $z=(z_1,z_2) \in C^2$, $z \neq 0$, $\alpha(z)^{-1}$ exists and $\alpha(z)^{-1} = (|z_1|^2 + |z_2|^2)^{-1} \alpha(z)$.

**Proof.** It is easy to see that $\alpha(zH) = [z]$, hence $\alpha(z)$ is invertible for any $z \neq 0$. Then

$$\alpha(z)^{-1} = (|z_1|^2 + |z_2|^2)^{-1} \alpha(z).$$

**Corollary 2.5.** For any $z \in C^2$, $z \neq 0$, we have

$$\| \alpha(z) \| = \| z \| \quad \text{and} \quad \| \alpha(z)^{-1} \| = \| z \|^{-1}$$

where $\| z \|^2 = |z_1|^2 + |z_2|^2$.

**Corollary 2.6.** If $a=(a_1,a_2)$ is nonsingular on $H$, then we have
the following commuting relations:

\[ a_1(a_1a_2 + a_2a_1)^* a_1 + a_1^*(a_1a_2 + a_2a_1) = I \]
(2.4)

\[ a_2(a_1a_2 + a_2a_1)^* a_2 + a_2^*(a_1a_2 + a_2a_1) = I \]

\[ a_1(a_1a_2 + a_2a_1)^* a_1 + a_1^*(a_1a_2 + a_2a_1) = 0. \]

Formulas (2.4) can be obtained by using the fact that (2.3) provides also a left inverse for \( \alpha(a) \).

3. Joint spectrum

**Lemma 3.1.** For a commuting pair of operators \( a = (a_1, a_2) \subseteq B(H) \), we have \( \sigma(a, H) = C^2 \setminus \{ z \in C^2 : (z - a)^t \subseteq B(H \oplus H) \} \).

**Corollary 3.2.** If \( a = (a_1, a_2) \subseteq B(H) \) is a commuting pair, then \( \alpha(a, H) = C^2 \setminus \{ z \in C^2 : (\alpha(z) - \alpha(a))^t \subseteq B(H \oplus H) \} \).

**Definition 3.3.** The mapping

\[ C^2 \setminus \alpha(a, H) \ni z \rightarrow R(z, a) = (\alpha(z) - \alpha(a))^t \subseteq B(H \oplus H) \]

is called the resolvent of \( a \).

**Lemma 3.4.** For a commuting pair \( a = (a_1, a_2) \subseteq B(H) \), the joint spectrum \( \sigma(a, H) \) is a closed set and the resolvent \( R(z, a) \) is an R-analytic function in \( C^2 \setminus \sigma(a, H) \).

**Proof.** Fix a point \( z_0 \notin \sigma(a, H) \). Since the map \( z \rightarrow \alpha(z) \) is isometric, then for \( z \in C^2 \) such that \( \| z - z_0 \| < \| \alpha(z_0 - a)^t \| \), the series

\[ \alpha(z_0 - a)^t \sum_{k=0}^{\infty} (-1)^k (\alpha(z_0) - \alpha(z_0 - a)^t)^k \]
is absolutely convergent and defines \((z-a)^i\). In particular, the set \(C^2 - \sigma(a,H)\) is open. Notice that \(\alpha(z-a)\) is a polynomial of degree one in \(z\) and \(z_k\), where we get easily that \(\alpha(z-a)^i\) is R-analytic in \(C^2 - \sigma(a,H)\).

**Lemma 3.5.** For a commuting pair \(a=(a_1,a_2)\subset B(H)\) and any \(z\) in \(C^2\) such that \(\|z\| > \|\alpha(a)\|\), we have \(z\) is not in \(\sigma(a,H)\) and

\[
(\alpha(z) - \alpha(a))^{-1} = \sum_{k=0}^{\infty} (\alpha(z)^{k} \alpha(a)^{k}) \alpha(z)^{-1}
\]

is absolutely and uniformly convergent on the sets \(\{z\in C^2 : \|z\| > r\}\) with \(r > \|\alpha(a)\|\).

**Proof.** According to Corollary 2.5, we have that if \(\|z\| > \|\alpha(a)\|\), then \(\|\alpha(a)^i \alpha(a)\| < 1\), hence the series (3.1) is absolutely convergent. It is straightforward to verify that (3.1) defines the inverse of \(\alpha(a)\). If \(r > \|\alpha(a)\|\), then for any \(z\) in \(C^2\) such that \(\|z\| > r\) we obtain by a direct estimation

\[
\| (\alpha(z) - \alpha(a))^{-1} \| \leq r(1 - \|\alpha(a)\|)^{-1},
\]

hence the convergence of (3.1) is uniform.

Notice that \(\lim_{k\to\infty} \| (\alpha(z) - \alpha(a))^{-1} \| = 0\).

**Theorem 3.6.** Let \(a=(a_1,a_2)\subset B(H)\) be a pair of commuting operators. Then the joint spectrum \(\sigma(a,H)\) of \(a\) is a compact nonempty set in \(C^2\).

**Proof.** On account of the Lemma 3.4 and Lemma 3.5, \(\sigma(a,H)\) is a compact subset of \(C^2\). Let us assume that \(\sigma(a,H)\) is empty. Then by Theorem 2.2 the operator

\[
((z_1-a_1)^* (z_1-a_1) + (z_2-a_2)^* (z_2-a_2))
\]
does exist, therefore the right ideal generated in $B(H)$ by $z - a$ is equal to $B(H)$ for any $z = (a, a_2) \in C^2$ which is, according to [3], a contradiction.

4. Applications

Let $H$ be a fixed Hilbert space. Let $a = (a_1, a_2)$ be a commuting pair of linear operators on $H$ and let $K$ be a closed subspace of $H$, $K$ reducing $a$, i.e. $aK \subset K$, $a_2K \subset K$ for $j = 1, 2$. We denote by $a|K$ the restrictions $(a_1|K, a_2|K)$.

**Proposition 4.1.** Assume that $a = (a_1, a_2) \subset B(H)$ is nonsingular on $H$ and $K$ be a closed subspace of $H$, $K$ reducing $a$. Then $a|K$ is nonsingular if and only if $a(a)^{-1}(K \oplus K) \subset K \oplus K$.

**Proof.** We apply Theorem 2.2 If $a|K$ is nonsingular, then $a(a|K)^{-1} \in B(K \oplus K)$. Take $\eta \in K \oplus K$. We have

$$a(a)(a(a)^{-1} \eta - a(a|K)^{-1} \eta) = 0,$$

hence $a(a)^{-1} \eta = a(a|K)^{-1} \eta \subset K \oplus K$. Conversely, if $a(a)^{-1}(K \oplus K) \subset K \oplus K$, then we have

$$a(a)^{-1} K = a(a|K)^{-1},$$

hence $a|K$ is nonsingular.

For any set $F \subset C^2$, let us denote by $aF$ the boundary of $F$.

**Proposition 4.2.** Let $K$ be a closed subspace of $H$, $K$ reducing $a$. Then we have the relation

$$\sigma(a|K) \subset \sigma(a, H).$$

**Proof.** Let us choose a point $z_0 \in \sigma(a|K)$ and suppose that $z_0 \notin \sigma(a, H)$. 

Then there is a sequence $z_k \in \sigma(a_1 K) \cap \sigma(a_2 H)$ such that $z_k \to z_0$ as $k \to \infty$. If $\eta \in K \oplus K$ is arbitrary, we can write

$$(a(z_0) - a(a))^{-1} \eta = \lim_{k \to \infty} (a(z_k) - a(a))^{-1} \eta \notin K \oplus K,$$

therefore $a(z_0) - a(a)$ is nonsingular on $K$, which is a contradiction.

**Corollary 4.3.** If $K_1$ and $K_2$ are closed subspaces, reducing $a$, such that $\sigma(a_1 K_1) \cap \sigma(a_2 K_2) = \phi$, then $K_1 \cap K_2 = 0$.

**Proof.** Indeed, $K_1 \cap K_2$ is reducing $a$, therefore

$$\sigma(a_1 K_1 \cap K_2) \subset \sigma(a_1 K_1) \cap \sigma(a_2 K_2) = \phi,$$

hence

$$\sigma(a_1 K_1 \cap K_2) = \phi, \text{ thus } K_1 \cap K_2 = 0.$$

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A Pair of Commuting Operators
on Hilbert Spaces


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