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MEASURES GENERATED BY DERIVATION BASES

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1. Introduction

In 1982, B.S. Thomson suggested the following question [4,p164] : If h is a monotonically increasing function on $[0,\infty)$, and $h(0)=0$, then the function $\tilde{h} : I \mapsto h(|I|)$ ($|I|$: the length of the interval I) represents a measure on R and $\tilde{h}_B$(B : Derivation bases, see definition 2.4) again represents a measure on R that should be related to classical p-dimensional Hausdorff measure. What is the exact relation here? About this question, S. Meinershagen in [2] defined three variational measures and he examined the relationship between these measures and Hausdorff measure. In this paper, we investigate some properties of these measures. Furhermore, we define the rarefaction indices of these measures and compare rarefaction index with Hausdorff dimension of some thin sets

2. Definitions and preliminaries

Let $H_0=\{h \mid h : [0,\infty)\to[0,\infty)$, monotone, increasing, continuous from the right, and $h(0)=0\}$ and let $B(R)$ be the family of bounded subsets of R.

Definition 2.1. ([2]) Let $h\in H_0$ and any set $E\subset[a,b]$. For any positive function $\delta : R\to R^+$, let $\beta_\delta^s=\{(I,x) : x$ is a midpoint of $I\subset(x-\delta(x), x+\delta(x))\}$.

$V(h^*, \beta_\delta^s[E]) = \sup\{\Sigma_{\pi[E]} h(|I_i|) : \pi = \{(I_i, x_i)\}$ is a partition of $[a,b]$ in β_δ^s and $(I_i, x_i) \in \pi[E]$ if $x_i \in E\}$. $V(h^*, D^s[E]) = \inf_\delta V(h^*, \beta_\delta^s[E])$. When $V(h^*, D^s[E])$ is being considered as a measure rather than a variation, it will be written by $h_S^*(E)$.

Definition 2.2. ([2]). Let h, E and δ be given as above and let $\beta_\delta = \{(I,x) : x$ is an end point of $I \subset (x - \delta(x), x + \delta(x))\}$. $V(h^*, \beta_\delta[E]) = \sup \{\Sigma_{\pi[E]} h(|I_i|) : \pi = \{(I_i, x_i)\}$ is a partition of $[a,b]$ in β_δ and $(I_i, x_i) \in \pi[E]$ if $x_i \in E\}$ and $h_D^*(E) = \inf_\delta V(h^*, \beta_\delta[E])$.

Definition 2.3. ([2]). Let h, E and δ be given as above and let $\beta_\delta^\# = \{(I,x) : I \subset (x-\delta(x), x+\delta(x))\}$. $V(h^*, \beta_\delta^\#[E]) = \sup\{\Sigma_{\pi[E]} h(|I_i|) : \pi = \{(I_i, x_i)\}$ is a partition of $[a,b]$ in $\beta_\delta^\#$ and $(I_i, x_i) \in \pi[E]$ if $x_i \in E\}$ and $h_{D\#}^*(E) = \inf_\delta V(h^*, \beta_\delta^\#[E])$.

Definition 2.4. Let $\beta = \{(I,x) : I$ and x are associated by a rule which determines how to choose the interval I in terms of the point $x\}$. A collection of such β's is called a derivation basis. For example, $B = \{\beta_\delta^s : \delta$ is any positive function$\}$ is a derivation basis which is called symmetric.

It is noted that $h_S^*(E) \le h_{D\#}^*(E)$, $h_D^*(E) \le h_{D\#}^*(E)$, and $h_S^*(E) \le h_D^*(E)$ if h is concave down. If $\bar{D}^+ h(0) = \infty$, then $h_{D\#}^*(E) = \infty$. So we dose not concern about $h_{D\#}^*$. In this paper, $h-m(E) = \lim_{\delta\to 0} h\text{-}m_\delta(E) = \liminf_{\delta\to 0} \{\Sigma h(|I_i|) : \cup I_i \supset E, |I_i| < \delta$ and I_i is an interval$\}$ denotes the Hausdorff measure. The rarefaction index corresponding to h-m is called Hausdorff dimension and is defined by $dim E = \inf\{a > 0 : e^a\text{-m}(E) = 0\} = \sup\{a > 0 : e^a\text{-}m(E) = \infty\}$, where if $h(x) = x^a$, then we write $h(x) = e^a$.

3. Results.

Let h^* be one of h_S^* and h_D^*.

Theorem 3.1. Let E be a bounded, h-m measurable subset of R and $0<h\text{-}m(E)<\infty$. Then $h\text{-}m(E)\leq h^*(E)$.

Proof. It is sufficient to prove that $h\text{-}m(F)\leq h_S^*(F)$ for a closed bounded subset F of E. Since h-m is regular, given $\varepsilon>0$, there exists a closed set $F\subset E$ such that $h\text{-}m(F)>h\text{-}m(E)\text{-}\varepsilon$. So $h\text{-}m(E)-\varepsilon<h-m(F)$ $\leq h_S^*(F)\leq h_S^*(E)\text{-}\varepsilon$.

Assume that $h_S^*(F)<\infty$ and $F\subset[a,b]$. Let δ be a positive function such that $(x\text{-}\delta(x),x+\delta(x))\subset[a,b]\setminus F$ if $x\notin F$(take $\delta(x)=\frac{\text{dist}(x,F)}{2}$ and $V(h^*,\beta_\delta^s[F])<h_S^*(F)+\varepsilon$.

Since $h\text{-}m(F)$ is finite, there exists $\delta_0>0$ such that $h\text{-}m(F)\text{-}\varepsilon<h\text{-}m_{\delta_0}(F)$. Put $\delta_1(x)=\min\{\delta(x),\frac{\delta_0}{2}\}$. Let $\pi\subset\beta_{\delta_1}^s$, where $\pi[F]=\{(I_i,x_i)\}_{i=1}^n$ and $x_i\in F$. Then $F\subset\cup_{i=1}^n I_i$. Therefore $h\text{-}m(F)\text{-}\varepsilon<h\text{-}m_{\delta_0}(F)\leq\Sigma_{i=1}^n h(\mid I_i\mid)\leq V(h^*,\beta_{\delta_1}^s(F))\leq V(h^*,\beta_\delta^s[F])<h_S^*(F)+\varepsilon$. Hence $h\text{-}m(F)\leq h_S^*(F)$.

We can use the similar methods for h_D^*.

Lemma 3.2. Let $h,g\in H_0$ and let $E\in B(R)$. If $\lim_{x\to 0}\frac{h(x)}{g(x)}=0$ and $g^*(E)<\infty$, then $h^*(E)=0$.

Proof. Let $g_D^*(E)=M<\infty$. There exists a positive function δ_1 such that $V(h^*,\beta_{\delta_1}[E])<M+1$. Since $\lim_{x\to 0}\frac{h(x)}{g(x)}=0$, for any $\varepsilon>0$, there exists $n_0\in N$ with $\frac{h(x)}{g(x)}>\frac{\varepsilon}{M+1}$ for all $0<x<\frac{1}{n_0}$. Put $\delta_2(x)=\min\{\delta_1(x),\frac{1}{n_0}\}$. For any positive function $\delta\leq\delta_2$, let $\pi=\{(I_i,x_i)\}_{i=1}^n$ be a partition of E in β_δ. Then $\Sigma_{i=1}^n h(\mid I_i\mid)<\frac{\varepsilon}{M+1}\Sigma_{i=1}^n g(\mid I_i\mid)\leq\frac{\varepsilon}{M+1}V(g^*,\beta_\delta[E])$

$\leq \frac{\varepsilon}{M+1} V(g^*, \beta_{\delta_1}[E]) < \varepsilon$. Hence $V(h^*, \beta_\delta[E]) < \varepsilon$ and $h_D^*(E)=0$.

Similar proof holds for h_S^*

If $h(x)=x^p$, then we denote $h^*(E)=h^p(E)$.

Theorem 3.3. There is a rarefaction index $\triangle$ such that for $E \in B(R)$,
(i) *If* $0<p<\triangle$, then $h^p(E)=\infty$ and
(ii) *If* $p>\triangle$, then $h^p(E)=0$.

Proof. Put $\triangle = \inf\{p>0 : h^p(E)=0\}$.
(i) Suppose that $h^p(E)<\infty$ for some $0<p<\triangle$. There exists p' such that $p<p'<\triangle$. By lemma 3.2, $h^{p'}(E)=0$. This is contradict to the definition of $\triangle$.
(ii) It is similar to (i).

So we can define the rarefaction index $\triangle$ of h^* like this ; $\triangle = \inf\{p>0 : h^p(E)=0\} = \sup\{p>0 : h^p(E)=\infty\}$. We have $\triangle=0$ if $h^p(E)=0$ for all $p>0$, $\triangle=\infty$ if $h^p(E)=\infty$ for all $p>0$. When h^* is h_S^* we write $\triangle=\triangle_S$, and when h^* is h_D^*, we write $\triangle=\triangle_D$

Remark.
(i) $\triangle(E_1) \leq \triangle(E_2)$ if $E_1 \subset E_2$.
(ii) $\triangle(E_1 \cup E_2) = \max\{\triangle(E_1), \triangle(E_2)\}$.
(iii) $dim(E) \leq \triangle$ by theorem 3.1.

Lemma 3.4. ([2], Corollary 4.2). *If* $\varliminf_{|I| \downarrow 0} \frac{h\text{-}m(E \cap I)}{h(|I|)} > 0$ (I is a symmetric interval about x) for every point x of E and $h\text{-}m(E)<\infty$, then $h^*(E)<\infty$.

There are sets with the same rarefaction index as Hausdorff dimension

Example 3.5(Cantor−like Set). Let $C=\{x\in R \mid x=\Sigma_{n=1}^{\infty}\frac{a_n}{4^n},\ a_n=0,3\}$. Then $\triangle_S=\frac{1}{2}$.

Proof. Let $h(x)=x^{\frac{1}{2}}$ and let I be any symmetric interval about $x\in C$. Let J be the largest contiguous interval of C in I. Then $I\subset 12J$ and $I\subset 48J'$, where J' is second large contiguous interval in J. So $h(\mid J'\mid)\leq h-m(C\cap I)$ (If $\mid J'\mid=\frac{2}{4^{n+1}}$, then $h-m(C\cap I)\leq\frac{1}{2^n}$). Therefore

$$\frac{h-m(C\cap I)}{h(\mid I\mid)}\geq\frac{h-m(C\cap I)}{h(48\mid J'\mid)}\geq\frac{h-m(C\cap I)}{h(48)h(\mid J'\mid)}\geq\frac{1}{h(48)}>0.$$

By lemma 3.4, $h_S^*(C)<\infty$. Hence $\triangle_S\leq\frac{1}{2}$. Since $h-m(C)=1\leq h_S^*(C)$, $\frac{1}{2}\leq\triangle_S$.

By analogous methods, we can obtain the rarefaction index of Cantor Ternary Set as $\frac{\log\ 2}{\log\ 3}$.

The following example shows that there is a set with different rarefaction index from Hausdorff dimension.

Example 3.6. Let $h(0)=h(0^+)=0$ and $D^+h(0)=\infty$. Then there is a compact set E such that $h_S^*(E)=\infty$ and $h\text{-}m(E)=0$. In particular, $h_S^{\alpha}(E)=\infty$ for all $0<\alpha<1$ and $e^{\alpha}\text{-}m(E)=0$ for all $0<\alpha<1$. Hence $\triangle_S=1$ and $\dim(E)=0$.

Construction. Let $p_0=\lambda_0=1$. $L_1^0=[0,1]$. We proceed by induction. Suppose that n is a natural number and that disjoint closed intervals $L_j^{n-1}(j=1,\ldots,p_{n-1})$ of length λ_{n-1} are given. Since $D^+h(0)=\infty$, there is

$q_n \in N$ such that $q_n h(\frac{\lambda_{n-1}}{2q_n}) > n$. Since $h(0^+)=0$, there is an $\eta_n \in (0, \frac{\lambda_{n-1}}{2q_n})$ such that $2q_n n p_{n-1} h(\eta_n) < 1$. For each i, let L_i^n be the closed interval of length η_n with the same center at J_i^n. Then $L_i^n \subset J_i^n$. Put $p_n = 2q_n p_{n-1}$, $\lambda_n = \eta_n$. Then $\lambda_n \leq 2^n$ for each n. Let $E = \cap_n (\cup_i L_i^n)$. Then $\Sigma_i h(|L_i^n|) = p_n h(\eta_n) < \frac{1}{n}$ for each n. Therefore $h\text{-}m(E)=0$. Furthermore, $h_S^*(E) = \infty$. [see[2], Example 4.2]

Suppose $f: I=[0,1] \rightarrow R$ and P is a partition of $[0,1]$. $0=x_0<x_1<\cdots<x_n=1$. $\sigma(p)=\max(x_i - x_{i-1})$. The limiting strong h-variation is defined by $V_h(f) = \lim_{\varepsilon \rightarrow 0} \sup_{\sigma(p)<\varepsilon} \Sigma_{i=1}^n h(|f(x_i)-f(x_{i-1})|)$.

Theorem 3.7. Suppose $f: I \rightarrow R$ continuous and let $h \in H_0$ be continuous with smoothness condition (there exists positive, finite c_0 such that $h(2x) < c_0 h(x)$ for $0<x<\frac{1}{2}$). Then $h_S^*(f(I)) \leq c_0 V_h(f)$

Proof. Given $\eta>0$, there exists ε such that $\sup_{\sigma(p)<\varepsilon} \Sigma_{i=1}^n h(|f(x_i)-f(x_{i-1})|) < V_h(f)+\eta$ and $\varepsilon^2 < h(s)$ for some $s>0$. Choose δ_0 such that $h(\delta_0)=\varepsilon^2$. Let δ be any positive function $\delta \leq \frac{\delta_0}{2}$ and let $\pi = \{(I_i, f(x_i))\}_{i=1}^m$ be a partition of $f(I)$ in β_δ^s. Together with 0 and 1, the points x_i, $1 \leq i \leq m$, form a partition P of $[0,1]$. Put $\pi_1 = \{I_i \in \pi \mid x_i - x_{i-1} \geq \varepsilon\}$ and $\pi_2 = \pi \setminus \pi_1$. Since $\Sigma(x_{i+1}-x_i) > \frac{1}{\varepsilon}\varepsilon = 1$, the number of π_1 does not exceed $\frac{1}{\varepsilon}$. So $\Sigma_{I_i \in \pi_1} h(|I_i|) \leq \frac{1}{\varepsilon} \cdot \varepsilon^2 = \varepsilon$. Add further division points to the dissection P_2(where $P_2 = \{x_i \mid (I_i, x_i) \in \pi_2\}$) to give a new partition P_3, with $\sigma(P_3)<\varepsilon$, containing no extra division point in (x_{i-1}, x_i) if $x_i \in P_2$. Since $|I_i| \leq 2|f(x_i)-f(x_{i-1})|$,

$$\sum_{I_i \in \pi_2} h(|I_i|) \leq \sum_{x_i \in P_2} h(2|f(x_i)-f(x_{i-1})|) \leq c_0 \sum_{x_i \in P_2} h(|f(x_i)-f(x_{i-1})|)$$

$$\leq c_0 \sum_{x_i \in P_3} h(|f(x_i)-f(x_{i-1})|) \leq c_0(V_h(f)+\eta).$$

Therefore $\Sigma_{n[f(I)]} h(|I_i|) \leq \varepsilon + c_0(V_h(f) + \eta)$.

This theorem is not sharp.

Example 3.8. Let $h(x)=x$ and define $f:[0,1]\rightarrow R$ by $f(0)=0$, $f(x)=x \sin \frac{1}{x}$ for $0<x\leq 1$. Then $V_h(f)=\infty$ and $h_S^*(f(I))\leq 2$.

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