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SOME PROPERTIES OF D^n-GROUPS

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1. Introduction

In 1980 and 1983, it was proved that PD^2-groups are surface groups ([2], [3]). Since then, topologists have been positively studying about PD^n-groups (or D^n-groups). For example, let a topological space X have a right π-action, where π is a multiplicative group. If each $x \in X$ has an open neighborhood U such that for each $u \in \pi$, $u \neq 1$, $U \cap U_u = \phi$, this right π-action is said to be proper. In this case, if X/π is compact then

(1) $\pi_1(X/\pi) \cong \pi$ (X : connected, π_1 : fundamental group) ([4]),

(2) if X is a differentiable orientable manifold with demension n and ∂X (the boundary of X) $= \phi$ then
$$H^k(X\,;\,Z) \cong H_{n-k}(X\,;\,Z), \quad ([6]),$$
where Z is the set of all integers.

In paticular, since
$R^n/(Z \times Z \times \cdots\cdots \times Z)$ (n-times) $\approx S^1 \times S^1 \times \cdots\cdots \times S^1$ (n-times) $= T^n$ is compact $Z \times Z \times \cdots \times Z$ (n-times) is a PD^n-group over Z (Theorem 3.1). Consider a short exact sequence of torsion free groups;
$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1.$$
Then, for a left G-module A we have the spectral sequence
$$E_2^{q,n} \cong H^q(Q\,;\,H^n(N, A)) \underset{q}{\Longrightarrow} H^{n+q}(G, A) \quad \text{(See 2)}.$$

Let K be the field of rationals. Our aim in this paper is to prove the following under suitable conditions:

(a) $E_2^{q,n+1} \cong E_2^{q+2,n}$.

(b) $0 \longrightarrow E_2^{q,n+r} \longrightarrow \cdots\cdots \longrightarrow E_2^{q+2r,n} \longrightarrow 0$

is exact (Theorem 3.3)

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2. Preliminaries

Let

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a short exact sequence of torsion free groups. Then, for a left G-module A we have a first quadrant spectral sequence $\{E_r, d_r\}$, natural in A with natural isomorphism

$$E_2^{p,q} \cong H^p(Q, H^q(N, A)) \underset{p}{\Longrightarrow} H^{p+q}(G, A). \qquad (*)$$

(Note that $E_2^{p,q} \underset{p}{\Longrightarrow} H^{p+q}(G\,;\,A)$ means $E_\infty^{p,q} \cong H^{p+q}(G, A)$) ([1], [7]).

Let us denote the cohomological dimension of G by cd(G).

Proposition 2.1. *In a short exact sequence of groups;*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

if $cd(N) = n < \infty$ *and* $cd(Q) = q < \infty$ *then* $cd(G) \leq n + q$.

Proof. For a left G-module A we have the spectral sequence

$$E_2^{p,q} = H^p(Q, H^q(N, A)) \underset{p}{\Longrightarrow} H^{p+q}(G, A)$$

as above $(*)$·

$$H^i(N, A) = 0 \text{ if } i > n \text{ and } H^j(Q, B) = 0 \text{ if } j > q,$$

where A is a left G-module and B is a left Q-module. Thus

$$E_2^{r,s} = H^r(Q, H^s(N, A)) = 0 \text{ if } r > q \text{ or } s > n,$$

and hence

$$E_\infty^{r,s} = H^{r+s}(G, A) = 0.$$

This means that for every left G-module A,

$$H^{r+s}(G, A) = 0 \text{ if } r + s > n + q$$

and thus cd$(G) \leq n + q \leq$ cd(N) + cd(Q).

Definition 2.2. A (multiplicative) group G is said to be of *type* (F, P) if there is a finite resolution over the trivial G-module Z by finitely generated projective modules, where Z is the set of integers.

Definition 2.3. A group G is a D^n-*group* over Z if and only if G satisfies the conditions

(1) G is of type (F, P)

(2) $H^k(G, Z[G]) = \begin{cases} 0, & \text{if } n \neq k \\ C, & \text{if } n = k, \end{cases}$

where $Z[G]$ is the group ring of G over Z.

(3) C is a Z-free module ([5], [8]).

A D^n-group sometimes is called a *Poincare duality group of dimension n.* Throughout this section, by a D^n-group we mean a D^n-group over Z without any statements.

Let G be a D^n-group. If $C=H^n(G, Z[G])\cong Z$ then G is said to be an *oriented Poincare duality group of dimension n,* written PD^n.

LEMMA 2. 4. *In the exact sequence of groups*

$$1\longrightarrow N\longrightarrow G\longrightarrow Q\longrightarrow 1,$$

if N is a D^n-group (PD^n-group) and Q is a D^q-group (PD^q-group) then G is a D^{n+q}-group(PD^{n+q}-group) such that

$$H^{n+q}(G, Z[G])\cong H^q(Q, Z[Q])\otimes H^n(N, Z[N]), \text{ where } \otimes=\otimes_Z.$$

Proof. By (*) above

$$E_2^{r,s}\cong H^r(Q, H^s(N, Z[G])\underset{p}{\Longrightarrow} H^{r+s}(G, Z[G])$$

on our case. Since

$$H^n(N, Z[G])\cong H^n(N, Z[N])\ \otimes_{Z[N]}Z[G]\ \ ([7], [9])$$

and

$$G/N=\{N, Nx_2, \cdots\},\quad Q\cong\{1, x_2, x_3, \cdots\}$$

is a group (N is normal in G) we have

$$H^n(N, Z[G])\cong H^n(N, Z[N])\ \otimes\ Z[Q],$$

and thus

$$E_2^{r,s}\cong H^r(Q, H^s(N, Z[N])\ \otimes\ Z[Q]).$$

By Definition 2. 3 and our assumption

$$H^k(N, Z[N])=0 \text{ if } k\neq n \text{ and } H^n(N, Z[N]) \text{ is } Z\text{-free},$$

and from

$$\begin{array}{ccccc} E_2^{r-2,n+1} & \longrightarrow & E_2^{r,n} & \longrightarrow & E_2^{r+2,n-1} \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$

we have $E_2^{r,n}=E_3^{r,n}=\cdots\cdots=E_\infty^{r,n}=H^{r+n}\ (G, Z[G])$.

Since Q is a D^q-group(PD^q-group)

$$\begin{aligned} &H^r(Q, H^n(N, Z[N])\ \otimes\ Z[Q] \\ &\quad\cong H_{q-r}(Q, H^q(Q, Z[Q])\otimes(Z[Q]\otimes H^n(N, Z[N]))) \\ &\quad\cong H_{q-r}(Q, H^q(Q, Z[Q])\otimes H^n(N, Z[N]))\ \ ([5], [9]), \end{aligned}$$

and thus $H^q(Q, H^n(N, Z[G]))\cong H_0(Q, H^q(Q, Z[Q])\otimes H^n(N, Z[N]))$

and

$$H^q(Q, H^n(N, Z[G])) \cong H^{n+q}(G, Z[G])$$
$$H_0(Q, H(Q, Z[Q])) \otimes H^n(N, Z[N]) \cong H^q(Q, Z[Q]) \otimes H^n(N, Z[N]).$$

That is,

$$H^k(G, Z[G]) = 0 \text{ if } n+q \neq k \text{ and}$$
$$H^{n+q}(G, Z[G]) \cong H^q(Q, Z[Q]) \otimes H^n(N, Z[N]).$$

Note that $H^n(N, Z[N])$ and $H^q(Q, Z[Q])$ are free Z-modules. Furthermore $H^{n+q}(G, Z[G])$ is Z-free.

Next, we have to prove that G is of type $(F.P)$. Since N is a D^n-group (PD^n-group) and Q is a D^q-group (PD^q-group), we have projective resolutions;

$$0 \longrightarrow P_n(N) \xrightarrow{\delta'} \cdots\cdots \longrightarrow P_0(N) \longrightarrow Z \longrightarrow 0$$
$$0 \longrightarrow P_q(Q) \xrightarrow{\delta''} \cdots \longrightarrow P_0(Q) \longrightarrow Z \longrightarrow 0,$$

where $P_i(N)$ $(i=0, \cdots, n)$ and $P_j(Q)$ $(j=0, \cdots, q)$ are finitely generated.

Put

$$P_k(G) = \sum_{i+j=k} (P_i(N) \otimes P_j(Q)),$$

then $P_k(G)$ is a finitely generated projective left $Z[G]$-module as follows. Since every element of G is represented by nx $(n \in N, x \in Q)$ by a unique way, if we define

$$nx(P_i(N) \otimes P_j(Q)) = (nP_i(N) \otimes (xP_j(Q))$$

then $P_i(N) \otimes P_j(Q)$ is a left G-module. In fact,

$$\begin{array}{ccc} Z[N] \otimes Z[Q] & \xrightarrow{\cong} & Z[G] \\ \Downarrow & & \Downarrow \\ (n \otimes x) & \longmapsto & nx \end{array}$$

is an isomorphism. For $n_1x_1,\ n_2x_2 \in G$, since

$$(n_1x_1)(n_2x_2) = (n_1n_2)(x_1x_2)$$

for $a \otimes b \in P_i(N) \otimes P_j(Q)$

$$(n_1x_1)(n_2x_2)(a \otimes b) = n_1n_2a \otimes x_1x_2b.$$

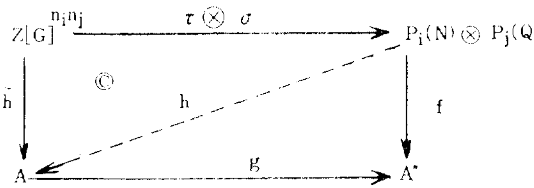
By our hypothesis there exist positive integers n_i, n_j, a $Z[N]$-epimorphism τ, and a $Z[Q]$-epimorphism σ such that

$$\tau : Z[N]^{n_i} \longrightarrow P_i(N)$$
$$\sigma : Z[Q]^{n_j} \longrightarrow P_j(Q).$$

Noting that $Z[N]$ and $X[Q]$ are Z-free we have isomorphisms

$$Z[N]^{n_i}\otimes Z[Q]^{n_j}\cong(Z(N)\otimes Z[Q])^{n_i n_j}\cong Z[G]^{n_i n_j}.$$

For a $Z[G]$-epimorphism $g: A\longrightarrow A''$ and a $Z[G]$-homomorphism $f: P_i(N)\otimes P_j(Q)\longrightarrow A''$, suppose the following diagram:



Then there exists a $Z[G]$-homomorphism $\tilde{h}$ such that $g\circ\tilde{h}=f\circ\tau\otimes\sigma$. Define $h: P_i(N)\otimes P_j(Q)\longrightarrow A$ by $h\circ(\tau\otimes\sigma)=\tilde{h}$. It follows that $g\circ h=f$. This means that each $P_i(N)\otimes P_j(Q)$ is a projective $Z[G]$-module. Therefore we have a projective semiexact sequence over Z as $Z[G]$-modules such that

$$P(G): 0\longrightarrow P_n(N)\otimes P_q(Q)\xrightarrow{\delta}\cdots\longrightarrow\sum_{i+j=k}P_i(N)\otimes P_j(Q)$$
$$\longrightarrow\cdots\longrightarrow P_0(N)\otimes P_0(Q)\longrightarrow Z\otimes Z=Z\longrightarrow 0,$$

where $\delta=\delta'\otimes\delta''$ and for $a\otimes b\in P_i(N)\otimes P_j(Q)$

$$(a\otimes b)=\delta'a\otimes b,\quad \delta''(a\otimes b)=(-1)^i a\otimes\delta''b.$$

For $a\otimes b\in P_i(N)\otimes P_j(Q)$ we assume that

$$\delta(a\otimes b)=\delta'a\otimes b+(-1)^i a\otimes\delta''b=0,$$

which is equivalent to $a\in\operatorname{Ker}\delta'$ and $b\in\operatorname{Ker}\delta''$. Thus there are $a'\in P_{j+1}(N)$, $b'\in P_{j+1}(Q)$ such that $\delta'a'=a$ and $\delta''b'=b$.

Thus, $a'\otimes b\in P_{i+1}(N)\otimes P_j(Q)$ and

$$\delta(a'\otimes b)=\delta'a'\otimes b+(-1)^{i+1}a'\otimes\delta''b.$$

Since $\delta''(b)=\delta''\delta''(b')=0$ and $\delta'a'=a$ we have $\delta(a'\otimes b)=a\otimes b$. That is, the complex $P(G)$ is exact. Consequently G is of type (F,P). If N and Q are PD-groups, then

$$H^n(N,Z[N])\cong Z\cong H^q(Q,Z[Q]),\ \text{and thus}$$
$$H^{n+q}(G,Z[G])\cong Z\otimes Z\cong Z.$$

3. Main theorems

From Lemma 2.4, we obtain the following:

THEOREM 3.1. *$Z\times Z\times\cdots\times Z$ (n-times) is a PD^n-group over Z.*

Proof. We denote each element $n \in Z$ by $[n]$ and define a multiplication by $[m][n]=[m+n]$. Therefore the group ring $Z[Z]$ is well defined. In $Z[Z]$

$$\sum n_i[a_i]+\sum m_i[a_i]=\sum (n_i+m_i)[a_i]$$

and

$$\sum n_i[a_i]\cdot \sum m_j[b_j]=\sum n_i m_j[a_i+b_j].$$

In particular, Z is the trivial $Z[Z]$-module, i.e., for $m \in Z$ and $n[a] \in Z[Z]$ $(n[a])m=nm$. Suppose a $Z[Z]$-module sequence.

$$(\sigma\text{-}\kappa):0\longrightarrow Z[Z]\xrightarrow{\sigma} Z[Z]\xrightarrow{\kappa} Z\longrightarrow 0,$$

is defined as follows:

$$\sigma([m])=[m]-[0]$$
$$\sigma(m_1+m_2)=[m_2]([m_1]-[-m_2])$$
$$([m]\sigma)([n])=[m]([n]-[-m]).$$

Then it follows that

$$m=n_1+n_2\Longrightarrow \sigma[m]=\sigma[n_1+n_2]$$
$$([m]\sigma)[n]=\sigma([m][n])=\sigma([m+n]).$$

Thus σ is a $Z[Z]$-monomorphism. κ is defined by $\kappa(n[m])=n$ for $n[m] \in Z[Z]$. Then κ is a $Z[Z]$-epimorphism. In particular, $\ker \kappa =\{\sum n_i[m_i]([a_i]-[0])\mid n_i[m_i],\ a_i \in Z[Z]\}$ and thus $\operatorname{Im}\sigma=\ker \kappa$. Therefore the sequence is exact as $Z[Z]$-modules. Hence the multiplicative group Z is of type (F, P). From the sequence $(\sigma-\kappa)$ we get the following.

$$0\longleftarrow \operatorname{Hom}_{Z[Z]}(Z[Z], Z[Z])\xleftarrow{\sigma^*}\operatorname{Hom}(Z[G], Z[G])\xleftarrow{K^*}$$
$$\operatorname{Hom}_{Z[Z]}(Z[Z], Z[Z])\cong Z[G],\qquad \operatorname{Hom}(Z[G], Z[G])\cong Z[G]$$
$$\operatorname{Hom}_{Z[Z]}(Z, [Z])\longrightarrow 0,\qquad \operatorname{Hom}_{Z[Z]}(Z, [Z])=0$$

Therefore,

$$H^k(Z, Z[Z])=\begin{cases} 0, & \text{if } k\neq 1\\ Z, & \text{if } k=1\end{cases}$$

and it follows that the multiplicative group Z is a PD^1-group over Z.

In general, $Z\times Z\times\cdots\times Z$ (n-times) is a multiplicative group with multiplication

$$([m_1],\cdots,[m_n])\cdot([n_1],\cdots,[n_n])=([m_1][n_1],\cdots,[m_n][n_n])$$

$$=[(m_1+n_1], \cdots, [m_n+n_n]).$$

We can use mathematical induction to complete our proof. Assume $Z\times Z\times, \cdots, \times Z((n-1)$-times) is a PD^{n-1}-group over Z. Consider the exact sequence

$$1\longrightarrow Z\times\cdots\times Z((n-1)\text{-times})\longrightarrow Z\times\cdots\times Z(n\text{-times})\longrightarrow Z\longrightarrow 1$$
$$([m_1], \cdots, [m_{n-1}])\longrightarrow([m_1], \cdots, [0])\longrightarrow[m_n]$$

Since $Z\times\cdots\times Z((n-1)$-times) is a PD^{n-1}-group over Z and Z is PD^1-group over Z by Lemma 2.4 $Z\times\cdots\times Z(n$-times) is a PD^n-group over Z.

Let us put K the set of all rationals. A D^n-group G over K is defined by

(1) G is of type (F, P) over K

(2) $H^i(G, K[G])$ is zero if $i\neq n$

(3) $H^n(G, K[G])$ is K-free, where $K[G]$ is the group ring of G over K.

Throughout this section we assume that every group is torsion free over K. The following is well-known

LEMMA 3.2. *If a group G is a D^n-group over Z, then G is also a D^n-group over K.* ([2], [3])

Let

$$1\longrightarrow N\longrightarrow G\longrightarrow Q\longrightarrow 1 \qquad (**)$$

be a short exact sequence of groups. Then by (*) in §2 we have the spectral sequence

$$E_2^{r,s}\cong H^r(Q, H^s(N, K[G]))\underset{r}{\Longrightarrow}H^{r+s}(G, K[G]).$$

Since $K[G]$ is $K[N]$-free

$$H^s(N, K[G])\cong H^s(N, K[N])\otimes K[Q],$$

and since every K-module is K-free $H^s(N, K[N])\otimes K[Q]$ is $K[Q]$-free. Thus we have the following ([6], [9]):

$$\begin{aligned}H^r(Q, H^s(N, K[G]))&\cong H^r(Q, H^s(N, K[N])\otimes_k K[Q])\\&\cong H^r(Q, K[Q])\otimes_{k[Q]}(H^s(N, K[N])\otimes_k K[Q])\\&\cong H^r(Q, K[Q])\otimes_k H^s(N, K[N])\\&\cong E_2^{r,s}.\end{aligned}$$

In (**) we assume that

(1) N is of type (F, P) over Z

(2) G is a D^{n+q}-group over Z
(3) cd $Z(Q)<\infty$.
Then we can prove that Q is of type (F, P) over Z ([9]).

THEOREM 3.3. *Under the above situation let us put*
$$m = \text{least number with } H^m(N, K[N]) \neq 0$$
$$p = \text{least number with } H^p(Q, K[Q]) \neq 0.$$
Then we have the following:
(1) $m=n$ *and* $p=q$
(2) *There is an isomorphism* $E_2^{q,n+1} \cong E_2^{q+2,n}$
(3) *There is an exact sequence*
$$0 \longrightarrow E_2^{q,n+r} \longrightarrow \cdots \longrightarrow E_2^{q+2r,n} \longrightarrow 0$$

Proof. (1) By the spectral sequence
$$E_2^{r,s} \cong H^r(Q, K[Q]) \otimes_k H^s(N, K[N])$$
and by our assumptions
$$E_2^{p,m}(\neq 0) \cong H^{p+m}(G, K[G]).$$
by Lemma 3.2 G is a D^{n+q}-group over K and thus $p=q$, $m=n$

(2) In our spectral sequence
$$\left.\begin{array}{ccccc} E_2^{q-2,n+2} & \longrightarrow & E_2^{q,n+1} & \xrightarrow[d_2]{} & E_2^{q+2,n} \\ \| & & & & \\ 0 & & & & \\ E_3^{q-3,n+3} & \longrightarrow & E_3^{q,n+1} & \longrightarrow & E_3^{q+2,n-1} \\ \| & & \| & & \| \\ 0 & & \ker d_2 & & 0 \end{array}\right\} \Rightarrow \begin{array}{l} E_3^{q,n+1} \cong \cdots \cong E_\infty^{q,n+1} \\ \cong H^{q+n+1}(G, K[G]) = 0 \end{array}$$
Thus $E_3^{q,n+1} = \operatorname{Ker} d_2 = 0$. From
$$\left.\begin{array}{ccccc} E_2^{q,n+1} & \xrightarrow[d_2]{} & E_2^{q+2,n} & \longrightarrow & E_2^{q+4,n-1} \\ & & & & \| \\ & & & & 0 \\ E_3^{q-1,n+2} & \longrightarrow & E_3^{q+2,n} & \longrightarrow & E_3^{q+5,n-2} \\ \| & & \| & & \| \\ 0 & & E_2^{d+2,n}/\operatorname{Im} d_2 & & 0 \end{array}\right\} \Rightarrow E_2^{q+2,n}/\operatorname{Im} d_2 = 0.$$
We have the isomorphism $d_2 : E_2^{q,n+1} \cong E_2^{q+2,n}$

(3) Since
$$\begin{array}{ccccc} E_2^{q-1,n+1} & \longrightarrow & E_2^{q+1,n} & \longrightarrow & E_2^{q+3,n-1} \\ \| & & & & \| \\ 0 & & & & 0 \end{array}$$
and $H^n(N, K[N]) \neq 0$, from
$$E_2^{q+1,n} \cong H^{q+1}(Q, K[Q]) \otimes_k H^n(N, K[N]) \cong E_\infty^{q+1,n}$$

$$\cong H^{q+1+n}(G, K[G])=0$$

$H^{q+1}(Q, K[Q])=0$. Similarly, from

$$0=E_2^{q+1,n+1}\longrightarrow E_2^{q+3,n}\longrightarrow E_2^{q+5,n-1}=0$$

$$H^{q+3}(Q, K[Q])=0.$$

Consequently

$$H^{q+i}(Q, K[Q])=0 \text{ for } i=1,3,5,\cdots.$$

Therefore we have

$$E_3^{q+s,n+t}=0$$

because that

(1) s : odd $\Longrightarrow E_2^{q+s,n+t}=0 \Longrightarrow E_3^{q+s,n+t}=0$

(2) s : even $\Longrightarrow \underset{\substack{\| \\ 0}}{E_3^{q+s-3,n+t+2}} \longrightarrow E_3^{q+s,n+t} \longrightarrow \underset{\substack{\| \\ 0}}{E_3^{q+s+3,n+t-2}}$

$$\Longrightarrow E_3^{q+s,n+t}=E_\infty^{q+s,n+t}=H^{q+s+n+t}(G, K[G])=0$$

According, we get the exact sequence

$$0\longrightarrow E_2^{q,n+r}\longrightarrow E_2^{q+2,n+r-1}\longrightarrow\cdots\longrightarrow E_2^{q+2r,n}\longrightarrow 0.$$

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