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A NONCOMMUTATIVE BUT INTERNAL MULTIPLICATION ON THE BANACH ALGEBRA A_t

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1. Introduction

In [1], Johnson and Lapidus introduced a family $\{A_t : t>0\}$ of Banach algebras of functionals on Wiener space and showed that for every F in A_t, the analytic operator-valued function space integral $K_\lambda^t(F)$ exists for all nonzero complex numbers λ with nonnegative real part. In [2, 3] Johnson and Lapidus introduced a noncommtative multiplication * having the property that if $F \in A_{t_1}$ and $G \in A_{t_2}$ then $F*G \in A_{t_1+t_2}$ and

$$K_\lambda^{t_1+t_2}(F*G) = K_\lambda^{t_1}(F) K_\lambda^{t_2}(G),$$

Note that for F, G in A_t, $F*G$ is not in A_t but rather is in A_{2t} and so the multiplication * is not internal to the Banach algebra A_t. In this paper we introduce an internal noncommutative multiplication $\otimes$ on A_t having the property that for F, G in A_t, $F \otimes G$ is in A_t and

$$K_\lambda^t(F \otimes G) = K_{2\lambda}^t(F) K_{2\lambda}^t(G)$$

for all nonzero λ with nonnegative real part. Thus $\otimes$ is an auxiliary binary operator on A_t.

2. Preliminaries

We will adopt much of the notation and terminology used in [1, 3]. However we will include a brief description of the Banach algebra A_t and the operator-valued function space integral K_λ^t(F). Let C, C_+ and $C_+^\sim$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part respectively. Let $L^2(\boldsymbol{R}^N)$ denote the space of

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Borel measurable, C-valued functions ϕ on $\mathbf{R}^N$ such that $|\phi|^2$ is integrable with respect to Lebesgue measure on $\mathbf{R}^N$.

For given $t>0$, let $C[0,t]$ denote the $\mathbf{R}^N$-valued continuous functions on $[0,t]$ and let $C_0[0,t]$ denote Wiener space; that is the set of all functions in $C[0,t]$ that vanish at 0. Let $m_{[0,t]}$ denote Wiener measure on $C_0[0,t]$.

Let $F : C[0,t]\to \mathbf{C}$ be Borel measurable. For given $\lambda>0, \phi\in L^2(\mathbf{R}^N)$, and $\xi\in\mathbf{R}^N$, consider the expression

$$(2.1)\quad (K_\lambda(F)\phi)(\xi)=\int_{C_0[0,t]} F(\lambda^{-1/2}x+\xi)\phi(\lambda^{-1/2}x(t)+\xi)dm_{[0,t]}(x).$$

The operator-valued function space integral $K_\lambda^a(F)$ exists for $\lambda>0$ if (2.1) defines $K_\lambda^a(F)$ as an element of $L(L_2(\mathbf{R}^N))$, the space of bounded linear operators on $L_2(\mathbf{R}^N)$. If, in addition, $K_\lambda^t(F)$, as a function of λ, has an extension to an analytic function on C_+ and to a strongly continuous function on $C_+^\sim$, we say that $K_\lambda^t(F)$ exists for $\lambda\in\mathbf{C}_+^\sim$. When λ is purely imaginary, $K_\lambda^t(F)$ is called the analytic operator-valued Feynman integral of F.

Let $M[0,t)$ denote the space of C-valued Borel measures on $[0,t)$. Given $\eta\in M[0,t)$, let $L_{\infty1:\eta}[0,t)$ denote the class of all Borel measurable functions $\theta : [0,t)\times\mathbf{R}^N\to\mathbf{C}$ such that

$$\|\theta\|_{\infty1:\eta}=\int_{[0,t)}\|\theta(s,\cdot)\|_\infty d|\eta|(s)<\infty.$$

A_t consists of all functions (actually equivalence classes of functions) on $C[0,t]$ of the form

$$F(x)=\sum_{n=0}^{\infty}\prod_{k=1}^{m_n}\int_{[0,t)}\theta_{n,k}(s,x(s))d\eta_{n,k}(s)$$

where

$$(2.2)\qquad \sum_{n=0}^{\infty}\prod_{k=1}^{m_n}\|\theta_{n,k}\|_{\infty1:\eta_{n,k}}<\infty.$$

For $F\in A_t$, let $\|F\|_t$ be the infimum of the left-hand side of (2.2) over all such representations of F. In [1, Theorem 6.1], Johnson and Lapidus show that $(A_t, \|\cdot\|_t)$ is a commutative Banach algebra under pointwise multiplication and addition. In addition they show that given $F\in A_t, K_\lambda^t(F)$ exists for all $\lambda\in\mathbf{C}_+^\sim$ and satisfies $\|K_\lambda(F)\|\le\|F\|_t$.

3. A lemma concerning Wiener measure

Let a and b be positive real numbers. Let $E_{a,b} : C[0,a] \to C[0,b]$ be given by the formula

$$E_{a,b}(x)(s) = \sqrt{\frac{b}{a}}\, x\left[\frac{as}{b}\right] \text{ for } 0 \leq s \leq b.$$

Then $E_{a,b}$ is bijective and continuous under the topology of uniform convergence.

Lemma 3.1. $m_{[0,a]} = m_{[0,b]} \circ E_{a,b}$.

Proof. It will suffice to show that $m_{[0,a]}(I) = m_{[0,b]}(E_{a,b}(I))$ where I is an arbitrary interval in $C_0[0,a]$. So let

$$I = \{x \in C_0[0,a] : (x(t_1), \cdots, x(t_n)) \in B\}$$

with $0 < t_1 < \cdots < t_n < a$ and B a Lebesgue measurable subset of $\boldsymbol{R}_n$. Then it is quite easy to see that

$$E_{a,b}(I) = \left\{ y \in C_0[0,b] : \left(y\left(\frac{bt_1}{a}\right), \cdots, y\left(\frac{bt_n}{a}\right)\right) \in \sqrt{\frac{b}{a}}B \right\}.$$

But by the definition of Wiener measure we see that

$$\begin{aligned} m_{[0,b]}(E_{a,b}(I)) &= \left\{ \prod_{k=0}^{n} \left[2\pi \frac{b}{a} (t_k - t_{k-1}) \right]^{-n/2} \right\} \cdot \\ &\quad \int_{\sqrt{\frac{b}{a}}B} \exp\left\{ -\frac{a}{2b} \sum_{k=1}^{n} \frac{|u_k - u_{k-1}|^2}{t_k - t_{k-1}} \right\} du_1 \cdots du_n \\ &= \left\{ \prod_{k=1}^{n} [2\pi (t_k - t_{k-1})]^{-n/2} \right\} \cdot \\ &\quad \int_B \exp\left\{ -\sum_{k=1}^{n} \frac{|v_k - v_{k-1}|^2}{2(t_k - t_{k-1})} \right\} dv_1 \cdots dv_n \\ &= m_{[0,a]}(I), \end{aligned}$$

where $t_0 = 0$, $u_0 = 0$, and $v_0 = 0$.

From the change of variable, we have following lemma.

Lemma 3.2. *Let f be a real or complex valued functional defined on $C_0[0,a]$. Then f is Wiener measurable on $C_0[0,a]$ if and only if $f \circ E_{a,b}^{-1}$ is Wiener measurable on $C_0[0,b]$. Furthermore,*

$$\int_{C_0[0,a]} f(x)\, dm_{[0,a]}(x) = \int_{C_0[0,b]} f\left(\sqrt{\frac{a}{b}}\, y\left(\frac{b}{a}(\cdot)\right)\right) dm_{[0,b]}(y), \tag{3.1}$$

where existence of one side implies that of the other and their equality.

4. A main result

In this section, for given F in A_a, we define a function $\bar{F}$ in $A_{a/2}$ and then show that $K_\lambda^a(F)=K_{\lambda/2}^{a/2}(\bar{F})$ for all λ in $C_+^\sim$.

Let F be a function in A_a. Then we can write F in the form

$$(4.1)\qquad F(x)=\sum_{n=0}^{\infty}\prod_{k=1}^{m_n}\int_{[0,a)}\theta_{n,k}(s,x(s))d\eta_{n,k}(s)$$

where each $\eta_{n,k}$ is in $M[0,a)$ and each $\theta_{n,k}$ is in $L_{\infty 1:\eta_{n,k}}$. Now for each n and k we define a measure $\bar{\eta}_{n,k}$ in $M\left[0,\frac{a}{2}\right)$ by the fomula

$$\bar{\eta}_{n,k}(B)=\eta_{n,k}(2B)$$

for each Borel subset B of $\left[0,\frac{a}{2}\right)$. we also define $\bar{\theta}_{n,k}$ in $L_{\infty 1:\bar{\eta}_{n,k}}$ by $\bar{\theta}_{n,k}(s,v)=\theta_{n,k}(2s,v)$ for all $(s,v)\in\left[0,\frac{a}{2}\right)\times R^N$. We note that $\|\eta_{n,k}\|=\|\bar{\eta}_{n,k}\|$ and $\|\bar{\theta}_{n,k}\|_{\infty 1:\bar{\eta}_{n,k}}=\|\theta_{n,k}\|_{\infty 1;\eta_{n,k}}$. Now we define $\bar{F}:C\left[0,\frac{a}{2}\right]\to C$ by

$$(4.2)\qquad \bar{F}(y)=\sum_{n=0}^{\infty}\prod_{k=1}^{m_n}\int_{\left[0,\frac{a}{2}\right)}\bar{\theta}_{n,k}(t,y(t))d\bar{\eta}_{n,k}(t).$$

It is quite easy to verify that $\bar{F}$ is in $A_{a/2}$ with $\|\bar{F}\|_{a/2}=\|F\|_a$.

THEOREM 4.1. *Let F in A_a be given by* (4.1) *and let $\bar{F}$ be given by* (4.2). *Then $K_\lambda^a(F)=K_{\lambda/2}^{a/2}(\bar{F})$ for all λ in $C_+^\sim$.*

Proof. Let $F_n(x)=\prod_{k=1}^{m_n}\int_{[0,a)}\theta_{n,k}(s,x(s))\,d\eta_{n,k}(s)$.

We will show that $K_\lambda^a(F_n)=K_{\lambda/2}^{a/2}(\bar{F}_n)$, The general case will then follow by use of the dominated convergence theorem.

First let $U_n:C_0[0,a]\times[0,a)^{m_n}\to C_0\left[0,\frac{a}{2}\right]\times\left[0,\frac{a}{2}\right)^{m_n}$ be defined by

$$\begin{aligned}U_n(x,s_1,\cdots,s_{m_n})&=\left(E_{a,a/2}(x),\tfrac{1}{2}s_1,\cdots,\tfrac{1}{2}s_{m_n}\right)\\&=\left(\frac{1}{\sqrt{2}}x(2\cdot),\frac{1}{2}s_1,\cdots,\frac{1}{2}s_{m_n}\right).\end{aligned}$$

Then by use of Lemma 3.1 it follows that

$$(4.3)\qquad m_{[0,a]}\times\eta_{n,k}=m_{[0,a/2]}\times\bar{\eta}_{n,k}\circ U_n.$$

Next let $\lambda>0$, $\psi\in L^2(R^N)$ and $\xi\in R^N$ be given. Then, we obtain the following equalities:

$$(K_{\lambda/2}^{a/2}(\bar{F}_n)\psi)(\xi)$$

$$\underset{=}{(1)} \int_{C_0[0,a/2]} \prod_{k=1}^{m_n} \int_{[0,a/2)} \bar{\theta}_{n,k}(t, (\lambda/2)^{-1/2}y(t)+\xi d\bar{\eta}_{n,k}(t) \psi((\lambda/2)^{-1/2}y(a/2)+\xi)dm_{[0,a/2]}(y)$$

$$\underset{=}{(2)} \int_{C_0[0,a/2]} \left[\prod_{k=1}^{m_n} \int_{[0,a]} \theta_{n,k}\left(s, \left(\frac{\lambda}{2}\right)^{-1/2} y\left(\frac{s}{2}\right)+\xi\right) d\eta_{n,k}(s)\right]\cdot \psi\left(\left(\frac{\lambda}{2}\right)^{-1/2} y\left(\frac{a}{2}\right)+\xi\right)dm_{[0,a/2]}(y)$$

$$\underset{=}{(3)} \int_{C_0[0,a]} \prod_{k=1}^{m_n} \int_{[0,a)} \theta_{n,k}(s, \lambda^{-1/2}x(s)+\xi)d\eta_{n,k}(s)\cdot \psi(\lambda^{-1/2}x(a)+\xi)dm_{[0,a]}(x)$$

$$\underset{=}{(4)} (K_\lambda^a(F_n)\psi)(\xi).$$

Equalities (1), and (4) follow by definition, (2) follows from the Lemma 3.2 and (3) follows from equation (4.3). Thus $K_\lambda^a(F_n)$ and $K_{\lambda/2}^{a/2}(\bar{F}_n)$ are equal for $\lambda>0$. But each is an analytic function of λ on C_+ and each is strongly continuous on $C_+^\sim$ which concludes the proof of the the theorem.

5. The multiplication operator $\otimes$

In this section we introduce the internal noncommuative multiplication $\otimes$ on A_a and then we proceed to obtain a formula for $K_\lambda^a(F\otimes G)$.

For $x\in C[0,a]$ let $R_1(x)$ be the restriction of x to $[0,a/2]$; that is to say $R_1 : C[0,a]\to C[0,a/2]$ is given by $R_1(x)(s)=x(s)$ for $0\le s\le a/2$. Also for $x\in C[0,a]$ let $R_2(x)$ be the restriction of x to $[0,a/2]$; that is to say $R_2 : C[0,a]\to C[a/2,a]$ is given by $R_2(x)(s)=x(s)$ for $a/2\le s\le a$. Also let $T : C[a/2,a]\to C[0,a/2]$ be the translation map

$$T(x)(s)=x(s+a/2),\quad 0\le s\le a/2.$$

Definition. For F and G in A_a, we define

$$F\otimes G : C[0,a]\to C$$

by the formula

$$(F\otimes G)(x)=\bar{F}(R_1(x))\bar{G}(T(R_2(x)) =(\bar{F}\circ R_1)(x)(\bar{G}\circ T\circ R_2)(x).$$

REMARK. In view of the definition of * given by equation (3.2) of [3] we see that $F \otimes G = \bar{F} * \bar{G}$.

THEOREM 5.1. *For F, G in A_a, $K_\lambda^a(F \otimes G) = K_{2\lambda}^a(F) K_{2\lambda}^a(G)$ for all λ in $C_+^\sim$.*

Proof. Let F and G be in A_a, Then by Theorem 4.1, $\bar{F}$ and $\bar{G}$ are in $A_{a/2}$ and we have that $K_{2\lambda}^a(F) = K_\lambda^{a/2}(\bar{F})$ and

$$K_{2\lambda}^a(G) = K_\lambda^{a/2}(\bar{G})$$

for all λ in $C_+^\sim$. But $F \otimes G = \bar{F} * \bar{G}$ and so by Theorem 5.3 of [3] if follows that

$$K_\lambda^a(\bar{F} * \bar{G}) = K_\lambda^{a/2}(\bar{F}) K_\lambda^{a/2}(\bar{G}).$$

Hence

$$\begin{aligned} K_\lambda^a(F \otimes G) &= K_\lambda^a(\bar{F} * \bar{G}) \\ &= K_\lambda^{a/2}(\bar{F}) K_\lambda^{a/2}(\bar{G}) \\ &= K_{2\lambda}^a(F) K_{2\lambda}^a(G). \end{aligned}$$

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