A NONCOMMUTATIVE BUT INTERNAL MULTIPLICATION ON THE BANACH ALGEBRA $A_t$

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1. Introduction

In [1], Johnson and Lapidus introduced a family $\{A_t : t > 0\}$ of Banach algebras of functionals on Wiener space and showed that for every $F$ in $A_t$, the analytic operator-valued function space integral $K_t^t(F)$ exists for all nonzero complex numbers $\lambda$ with nonnegative real part. In [2, 3] Johnson and Lapidus introduced a noncommutative multiplication $*$ having the property that if $F \in A_t$, and $G \in A_t$, then $F*G \in A_{t+t_t}$, and

$$K_{t+t_t}^t(F*G) = K_t^t(F)K_t^t(G),$$

Note that for $F, G$ in $A_t$, $F*G$ is not in $A_t$ but rather is in $A_{t+t_t}$ and so the multiplication $*$ is not internal to the Banach algebra $A_t$. In this paper we introduce an internal noncommutative multiplication $\otimes$ on $A_t$ having the property that for $F, G$ in $A_t$, $F \otimes G$ is in $A_t$ and

$$K_t^t(F \otimes G) = K_{t+t_t}^t(F)K_{t+t_t}^t(G)$$

for all nonzero $\lambda$ with nonnegative real part. Thus $\otimes$ is an auxiliary binary operator on $A_t$.

2. Preliminaries

We will adopt much of the notation and terminology used in [1, 3]. However we will include a brief description of the Banach algebra $A_t$ and the operator-valued function space integral $K_t^t(F)$. Let $\mathbb{C}, \mathbb{C}_+$ and $\mathbb{C}_+$ denote the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real parts respectively. Let $L^2(\mathbb{R}^t)$ denote the space of

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Borel measurable, $C$-valued functions $\phi$ on $\mathbb{R}^n$ such that $|\phi|^2$ is integrable with respect to Lebesgue measure on $\mathbb{R}^n$.

For given $t>0$, let $C[0,t]$ denote the $\mathbb{R}^n$-valued continuous functions on $[0,t]$ and let $C_0[0,t]$ denote Wiener space; that is the set of all functions in $C[0,t]$ that vanish at 0. Let $m_{[0,t]}$ denote Wiener measure on $C_0[0,t]$. Let $F: C[0,t] \to \mathbb{C}$ be Borel measurable. For given $\lambda>0, \phi \in L^2(\mathbb{R}^n)$, and $\xi \in \mathbb{R}^n$, consider the expression

$$(2.1) \quad (K_\lambda(F)\phi)(\xi) = \int_{C(0,t)} F(\lambda^{-1/2}x + \xi)\phi(\lambda^{-1/2}x(t) + \xi)dm_{[0,t]}(x).$$

The operator-valued function space integral $K_\lambda(F)$ exists for $\lambda>0$ if (2.1) defines $K_\lambda(F)$ as an element of $L(L_2(\mathbb{R}^n))$, the space of bounded linear operators on $L_2(\mathbb{R}^n)$. If, in addition, $K_\lambda(F)$, as a function of $\lambda$, has an extension to an analytic function on $C_+$ and to a strongly continuous function on $C_\infty$, we say that $K_\lambda(F)$ exists for $\lambda \in C_\infty$. When $\lambda$ is purely imaginary, $K_\lambda(F)$ is called the analytic operator-valued Feynman integral of $F$.

Let $M[0,t]$ denote the space of $C$-valued Borel measures on $[0,t]$. Given $\eta \in M[0,t]$, let $L_{\infty,1}: M[0,t] \to C$ denote the class of all Borel measurable functions $\theta: [0,t] \times \mathbb{R}^n \to \mathbb{C}$ such that

$$\|\theta\|_{\infty,1} := \int_{[0,t]} \|\theta(s, \cdot)\|_{\infty} d|\eta|(s) < \infty.$$ 

$A_\infty$ consists of all functions (actually equivalence classes of functions) on $C[0,t]$ of the form

$$F(x) = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \int_{[0,t]} \int_{\mathbb{R}^n} (s, x(s))d\eta_{n,k}(s)$$

where

$$(2.2) \quad \sum_{n=0}^{\infty} \prod_{k=1}^{n} \|\theta_{n,k}\|_{\infty,1} < \infty.$$ 

For $F \in A_\infty$, let $\|F\|_i$ be the infimum of the left-hand side of (2.2) over all such representations of $F$. In [1, Theorem 6.1], Johnson and Lapidus show that $(A_\infty, \|\cdot\|_i)$ is a commutative Banach algebra under pointwise multiplication and addition. In addition they show that given $F \in A_\infty, K_\lambda(F)$ exists for all $\lambda \in C_+$ and satisfies $\|K_\lambda(F)\|_i \leq \|F\|_i$. 

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3. A lemma concerning Wiener measure

Let $a$ and $b$ be positive real numbers. Let $E_{a,b} : C[0, a] \to C[0, b]$ be given by the formula

$$E_{a,b}(x)(s) = \sqrt{\frac{b}{a}} x\left(\frac{as}{b}\right) \text{ for } 0 \leq s \leq b.$$ 

Then $E_{a,b}$ is bijective and continuous under the topology of uniform convergence.

**Lemma 3.1.** $m_{[0,a]} = m_{[0,b]} \circ E_{a,b}$. 

**Proof.** It will suffice to show that $m_{[0,a]}(I) = m_{[0,b]}(E_{a,b}(I))$ where $I$ is an arbitrary interval in $C_0[0, a]$. So let

$$I = \{ x \in C_0[0, a] : (x(t_1), \ldots, x(t_n)) \in B \}$$

with $0 < t_1 < \ldots < t_n < a$ and $B$ a Lebesgue measurable subset of $\mathbb{R}^n$. Then it is quite easy to see that

$$E_{a,b}(I) = \left\{ y \in C_0[0, b] : \left( y\left(\frac{bt_1}{a}\right), \ldots, y\left(\frac{bt_n}{a}\right) \right) \in \sqrt{\frac{b}{a}} B \right\}.$$ 

But by the definition of Wiener measure we see that

$$m_{[0,b]}(E_{a,b}(I)) = \left\{ \prod_{k=0}^{n} \left[ 2\pi \frac{b}{a} (t_k - t_{k-1}) \right]^{-n/2} \right\} \cdot$$

$$\int \sqrt{\frac{b}{a}} \exp\left\{ -\frac{a}{2b} \sum_{k=1}^{n} \frac{|u_k - u_{k-1}|^2}{t_k - t_{k-1}} \right\} du_1 \cdots du_n$$

$$= \left\{ \prod_{k=1}^{n} \left[ 2\pi (t_k - t_{k-1}) \right]^{-n/2} \right\} \cdot$$

$$\int_B \exp\left\{ -\frac{1}{2(t_k - t_{k-1})} \right\}dv_1 \cdots dv_n$$

$$= m_{[0,a]}(I),$$

where $t_0 = 0$, $u_0 = 0$, and $v_0 = 0$.

From the change of variable, we have following lemma.

**Lemma 3.2.** Let $f$ be a real or complex valued functional defined on $C_0[0, a]$. Then $f$ is Wiener measurable on $C_0[0, a]$ if and only if $f \circ E_{a,b}^{-1}$ is Wiener measurable on $C_0[0, b]$. Furthermore,

$$\int_{C_0[0,a]} f(x) dm_{[0,a]}(x) = \int_{C_0[0,b]} f\left( \sqrt{\frac{b}{a}} y\left(\frac{b}{a} (\cdot) \right) \right) dm_{[0,b]}(y),$$

where existence of one side implies that of the other and their equality.
4. A main result

In this section, for given $F$ in $A$, we define a function $\bar{F}$ in
$A_{\alpha/2}$ and then show that $K_{\lambda}^{\bar{F}}(F) = K_{\lambda/2}^{\bar{F}}(F)$ for all $\lambda$ in $C^{\infty}$.

Let $F$ be a function in $A$. Then we can write $F$ in the form

\begin{equation}
F(x) = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \int_{[0,a)} \theta_{n,k}(s, x(s)) \, d\eta_{n,k}(s)
\end{equation}

where each $\eta_{n,k}$ is in $M[0,a)$ and each $\theta_{n,k}$ is in $L_{\infty; \gamma_{n,k}}$. Now for
each $n$ and $k$ we define a measure $\bar{\eta}_{n,k}$ in $M\left[0, \frac{a}{2}\right]$ by the formula

$\bar{\eta}_{n,k}(B) = \eta_{n,k}(2B)$

for each Borel subset $B$ of $[0, \frac{a}{2})$. we also define $\bar{\theta}_{n,k}$ in $L_{\infty; \gamma_{n,k}}$ by

$\bar{\theta}_{n,k}(s, v) = \theta_{n,k}(2s, v)$ for all $(s, v) \in \left[0, \frac{a}{2}\right] \times R^n$. We note that $||\eta_{n,k}|| = ||\bar{\eta}_{n,k}||$ and $||\bar{\theta}_{n,k}||_{\infty; \gamma_{n,k}} = ||\theta_{n,k}||_{\infty; \gamma_{n,k}}$. Now we define $F : C\left[0, \frac{a}{2}\right] \to C$ by

\begin{equation}
F(y) = \sum_{n=0}^{\infty} \prod_{k=1}^{n} \int_{[0,a)} \bar{\theta}_{n,k}(t, y(t)) \, d\bar{\eta}_{n,k}(t).
\end{equation}

It is quite easy to verify that $\bar{F}$ is in $A_{\alpha/2}$ with $||\bar{F}||_{\alpha/2} = ||F||_{\alpha}$.

**Theorem 4.1.** Let $F$ in $A$ be given by $(4.1)$ and let $\bar{F}$ be given
by $(4.2)$. Then $K_{\lambda}^{\bar{F}}(F) = K_{\lambda/2}^{\bar{F}}(F)$ for all $\lambda$ in $C^{\infty}$.

**Proof.** Let $F_n(x) = \prod_{k=1}^{n} \theta_{n,k}(s, x(s)) \, d\eta_{n,k}(s)$.

We will show that $K_{\lambda}^{\bar{F}}(F_n) = K_{\lambda/2}^{\bar{F}}(F_n)$, The general case will then
follow by use of the dominated convergence theorem.

First let $U_n : C_0[0, a] \times [0, a]^{m_n} \to C_0\left[0, \frac{a}{2}\right] \times \left[0, \frac{a}{2}\right]^{m_n}$ be defined by

\begin{align*}
U_n(x, s_1, \ldots, s_{m_n}) &= (E_{\alpha/2}(x), \frac{1}{2}s_1, \ldots, \frac{1}{2}s_{m_n}) \\
&= \left(\frac{1}{\sqrt{2}}x(2 \cdot), \frac{1}{2}s_1, \ldots, \frac{1}{2}s_{m_n}\right).
\end{align*}

Then by use of Lemma 3.1 it follows that

\begin{equation}
m_{[0,a]} \times \eta_{n,k} = m_{[0,a/2]} \times \bar{\eta}_{n,k} \circ U_n.
\end{equation}

Next let $\lambda > 0$, $\phi \in L^2(R^n)$ and $\xi \in R^n$ be given. Then, we obtain
the following equalities:
A noncommutative but internal multiplication on the Banach algebra $A$,

$$(K_2^\infty F_n)\phi)(\xi)$$

(1) \[
\int_{C[0,a/2]} \prod_{k=1}^{m_n} \int_{[0,a/2]} \bar{\theta}_{n,k}(t, (\lambda/2)^{-1/2}y(t) + \xi d\eta_{n,k}(t) \\
\phi((\lambda/2)^{-1/2}y(a/2) + \xi) dm_{[0,a/2]}(y)
\]

(2) \[
\int_{C[0,a/2]} \prod_{k=1}^{m_n} \int_{[0,a]} \theta_{n,k}(s, (\lambda/2)^{-1/2}y(\frac{s}{2}) + \xi) d\eta_{n,k}(s) \\
\phi((\lambda/2)^{-1/2}y(\frac{a}{2}) + \xi) dm_{[0,a/2]}(y)
\]

(3) \[
\int_{C[0,a]} \prod_{k=1}^{m_n} \int_{[0,a]} \theta_{n,k}(s, \lambda^{-1/2}x(s) + \xi) d\eta_{n,k}(s) \\
\phi(\lambda^{-1/2}x(a) + \xi) dm_{[0,a]}(x)
\]

(4) \[
(K_2^\infty (F_n)\phi)(\xi).
\]

Equalities (1), (2), and (4) follow by definition, (2) follows from the Lemma 3.2 and (3) follows from equation (4.3). Thus $K_2^\infty (F_n)$ and $K_2^\infty (F_n)$ are equal for $\lambda > 0$. But each is an analytic function of $\lambda$ on $C_+$ and each is strongly continuous on $C_+$ which concludes the proof of the theorem.

5. The multiplication operator $\otimes$

In this section we introduce the internal noncommutative multiplication $\otimes$ on $A_\alpha$ and then we proceed to obtain a formula for $K_2^\infty (F \otimes G)$.

For $x \in C[0,a]$ let $R_1(x)$ be the restriction of $x$ to $[0,a/2]$; that is to say $R_1 : C[0,a] \to C[0,a/2]$ is given by $R_1(x)(s) = x(s)$ for $0 \leq s \leq a/2$. Also for $x \in C[0,a]$ let $R_2(x)$ be the restriction of $x$ to $[0,a/2]$; that is to say $R_2 : C[0,a] \to C[a/2,a]$ is given by $R_2(x)(s) = x(s)$ for $a/2 \leq s \leq a$. Also let $T : C[a/2,a] \to C[0,a/2]$ be the translation map $T(x)(s) = x(s+a/2), 0 \leq s \leq a/2$.

**Definition.** For $F$ and $G$ in $A_\alpha$, we define $F \otimes G : C[0,a] \to C$ by the formula

$$(F \otimes G)(x) = \bar{F}(R_1(x)) \bar{G}(T(R_2(x)) = (\bar{F} \circ R_1)(x)(\bar{G} \circ T \circ R_2)(x).$$
Remark. In view of the definition of $*$ given by equation (3.2) of [3] we see that $F \otimes G = \bar{F}^* \bar{G}$.

Theorem 5.1. For $F, G$ in $A_\gamma$, $K^a_\gamma(F \otimes G) = K^a_{2\lambda}(F) K^a_{2\lambda}(G)$ for all $\lambda$ in $C_\gamma$.

Proof. Let $F$ and $G$ be in $A_\gamma$. Then by Theorem 4.1, $\bar{F}$ and $\bar{G}$ are in $A_{\gamma/2}$ and we have that $K^a_{2\lambda}(F) = K^a_{\gamma/2}(\bar{F})$ and

$$K^a_{2\lambda}(G) = K^a_{\gamma/2}(\bar{G})$$

for all $\lambda$ in $C_\gamma$. But $F \otimes G = \bar{F}^* \bar{G}$ and so by Theorem 5.3 of [3] if follows that

$$K^a_\gamma(\bar{F}^* \bar{G}) = K^a_{\gamma/2}(\bar{F}) K^a_{\gamma/2}(\bar{G}).$$

Hence

$$K^a_\gamma(F \otimes G) = K^a_\gamma(\bar{F}^* \bar{G}) = K^a_{\gamma/2}(\bar{F}) K^a_{\gamma/2}(\bar{G}) = K^a_{2\lambda}(F) K^a_{2\lambda}(G).$$

References


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