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WEAK CONVERGENCE FOR A REVERSIBLE SEMIGROUP OF LIPSCHITZIAN MAPPINGS

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1. Introduction

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a\in S$ the mappings $s\to a\cdot s$ and $s\to s\cdot a$ from S to S are continuous. S is called right reversible if any two closed left ideals of S has nonvoid intersection. In this case, $(S,\geq)$ is a directed system when the binary relation "$\geq$" on S is defined by

$$t\geq s \text{ if and only if } s\cup\overline{Ss}\supseteq t\cup\overline{St},\quad s,t\in S.$$

Left reversibility of S is defined similarly. S is called reversible if it is both left and right reversible.

In [4], Ishihara-Takahashi considers a semigroup of lipschitzian mappings: let C be a closed convex subset of a real Banach space X with norm $\|\cdot\|$. Then a family $\mathfrak{F}=\{T_s : s\in S\}$ of mappings from C into itself is said to be a *lipschitzian semigroup* on C *if* $\mathfrak{F}$ satisfies the following:

(a) the index set S is a semitopological semigroup;

(b) $T_{st}x=T_sT_tx$ for all $s,t\in S$ and $x\in C$;

(c) the mapping $(s,x)\to T_sx$ from $S\times C$ into C is continuous when $S\times C$ has the product topology;

(d) for each $s\in S$, there exists $k_s>0$ such that

$$\|T_sx-T_sy\|\leq k_s\|x-y\| \text{ for all } x,y\in C.$$

A lipschitzian semigroup $\mathfrak{F}=\{T_s : s\in S\}$ is called *reversible* [resp., *right*(*left*) *reversible*] if S is reversible [resp., right (left) reversible]. It is said to be *nonexpansive* if $k_s=1$ for all $s\in S$ and *asymptotically nonexpansive* if $\lim_{s\in S} k_s=1$, respectively. In case that $S=N$, where N denotes the set of natural numbers, T is said to be a *lipschitzian mapping*, written by $T_n=T^n$ for all $n\in N$.

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For each $x \in C$, $\mathfrak{I}(x) = \{T_s x : s \in S\}$ is called the *orbit* of x under $\mathfrak{I}$ and a point $z \in C$ such that $\mathfrak{I}(z) = \{z\}$ is called a *common fixed point* of $\mathfrak{I}$. We denote by $F(\mathfrak{I})$ the set of common fixed points of $\mathfrak{I}$.

Unless other specified, we suppose that C is a closed convex subset of a real Banach space X and $\mathfrak{I} = \{T_s : s \in S\}$ is a lipschitzian semigroup on C with $\limsup_{s \in S} k_s \leq 1$. Then we shall establish the weak convergence of the lipschitzian semigroup $\mathfrak{I} = \{T_s : s \in S\}$.

2. Weak convergence

For each $x \in C$, we set

$$E(x) = \{y \in C : \lim_{s \in S} \|T_s x - y\| \text{ exists}\}.$$

Then we begin with the following:

LEMMA 2.1. *For each $x \in C$, $F(\mathfrak{I}) \subseteq E(x)$.*

Proof. Let $y \in F(\mathfrak{I})$ and $r = \inf_{s \in S} \|T_s x - y\|$. Given $\varepsilon > 0$, there is $s_0 \in S$ such that $\|T s_0 x - y\| < r + \varepsilon$. Let $t \geq s_0$. Since S is right reversible, we may assume $t \in \overline{S s_0}$. Let $\{s_\alpha\}$ be a net in S such that $s_\alpha s_0 \to t$. Then for each α,

$$\begin{aligned} \|T_{s_\alpha s_0} x - y\| &= \|T_{s_\alpha} T_{s_0} x - T_{s_\alpha} y\| \\ &\leq k_{s_\alpha} \|T_{s_0} x - y\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|T_t x - y\| &\leq (\limsup_\alpha k_{s_\alpha}) \|T_{s_0} x - y\| \\ &\leq \|T_{s_0} x - y\|. \end{aligned}$$

So we have

$$\begin{aligned} \inf_{s \in S} \sup_{t \succ s} \|T_t x - y\| &\leq \sup_{t \succ s_0} \|T_t x - y\| \\ &\leq \|T_{s_0} x - y\| < r + \varepsilon. \end{aligned}$$

Since ϵ is arbitrary, we have

$$\inf_{s \in S} \sup_{t \succ s} \|T_t x - y\| \leq r = \inf_{s \in S} \|T_s x - y\|$$

Therefore, $\lim_s \|T_s x - y\|$ exists and so $y \in E(x)$.

When $\{x_\alpha\}$ is a net in a Banach space X and $x \in X$, $x_\alpha \to x$ ($x_\alpha \rightharpoonup x$) means the strong(weak) convergence to x of the net $\{x_\alpha\}$, respectively.

Recall that a Banach space X is said to satisfy Opial's condition if, for any net $\{x_\alpha\}$ in X with $x_\alpha \rightharpoonup x \in X$,

(*) $$\limsup_\alpha \|x_\alpha - x\| < \limsup_\alpha \|x_\alpha - y\|, \quad y(\neq x) \in X.$$

(see [8, Lemma 1] and [6, Lemma 2.1]). For more details, see also [3] and [7].

For each $x \in C$, we shall denote by $\omega_w(x)$ the set of weak subnet limits of $\{T_s x : s \in S\}$. By the slight modification of the proof of Theorem 1 in [1], we obtain the following:

LEMMA 2.2. *Let X satisfy Opial's condition and $x \in C$. If $\phi \neq \omega_w(x) \subseteq E(x)$, then the orbit $\{T_s x : s \in S\}$ converges weakly.*

We are now ready to prove the weak convergence of the orbit $\{T_s x : s \in S\}$, for each $x \in C$.

THEOREM 2.3. *Let X be uniformly convex and satisfy Opial's condition and $x \in C$. Then, the orbit $\{T_s x : s \in S\}$ converges weakly to an element of $F(\mathfrak{J})$ if and only if $F(\mathfrak{J}) \neq \phi$ and $T_{ts}x - T_s x \to 0$ fall all $t \in S$.*

Proof. We need only to prove the "if" part. Since $F(\mathfrak{J}) \neq \phi$, by Lemma 2.1, $\{T_s x : s \in S\}$ is bounded; hence $\omega_w(x) \neq \phi$. By Lemma 2.1 and Lemma 2.2, it suffices to show that $\omega_w(x) \subseteq F(\mathfrak{J})$. To this end, let $y \in \omega_w(x)$; hence there is a subset $\{T_{s_\alpha} x\}$ of the net $\{T_s x : s \in S\}$ for which $T_{s_\alpha} x \rightharpoonup y$ for all $t \in S$. Suppose that $y \notin F(\mathfrak{J})$ and set

$$r_t = \limsup_\alpha \| T_{ts_\alpha} x - y \|.$$

With a proof as in Lemma 2.1, we can see that $r = \lim_{t \in S} r_t = \inf_{t \in S} r_t$. Since $y \notin F(\mathfrak{J})$, it follows easily that $r > 0$.

For any fixed $\eta > 0$, choose $\varepsilon > 0$ so small that

$$(r+\varepsilon)\left[1 - \delta\left(\frac{\eta}{r+\varepsilon}\right)\right] < r,$$

where δ is the modulus of convexity of the norm. For given $\varepsilon > 0$, there is $s_0 \in S$ such that $r_s < r + \varepsilon/2$, for all $s \geq s_0$. Since $\limsup_{s \in S} k_s \leq 1$, there is also $t_0 \in S$ such that $k_t \leq 1 + \varepsilon/(2M)$ for all $t \geq t_0$, where

$M=\sup_{s\in S}\|T_s x-y\|<\infty$. Then, since $(S,\geq)$ is a directed system, there exists $a_0\in S$ with $a_0\geq s_0$ and $a_0\geq t_0$. Hence, for each α and for all $t\geq a_0$,

$$\begin{aligned}\|T_t\| T_{a_0 s_\alpha}x-T_t y\| &\leq k_t\|T_{a_0 s_\alpha}x-y\|\\ &\leq\left(1+\frac{\varepsilon}{2M}\right)\|T_{a_0 s_\alpha}x-y\|\\ &\leq\|T_{a_0 s_\alpha}x-y\|+\frac{\varepsilon}{2}.\end{aligned}$$

So, we have both

$$\lim_\alpha\sup\|T_{ta_0}T_{s_\alpha}x-T_t y\|<r+\varepsilon,$$

and

$$\lim_\alpha\sup\|T_{ta_0}T_{s_\alpha}x-y\|<r+\varepsilon,\ \text{for all } t\geq a_0.$$

Since $y\notin F(\mathfrak{I})$, we choose a $b\geq a_0$ with $T_b y\neq y$. Then, there is $s_{\alpha_0}\in S$ such that, for all $s_\alpha\geq s_{\alpha_0}$,

$$\|T_{ba_0}T_{s_\alpha}x-T_b y\|<r+\varepsilon,$$

and

$$\|T_{ba_0}T_{s_\alpha}x-y\|<r+\varepsilon.$$

It follows, by uniform convexity of X, that

$$\begin{aligned}\|T_{ba_0}T_{s_\alpha}x-\frac{1}{2}(T_b y+y)\| &\leq(r+\varepsilon)\left[1-\delta\left(\frac{\|T_b y-y\|}{r+\varepsilon}\right)\right]\\ &<r \text{ for all } s_\alpha\geq s_{\alpha_0}.\end{aligned}$$

Then, Opial's condition(*) implies that

$$\begin{aligned}r_{ba_0}&<\lim_\alpha\sup\|T_{ba_0}T_{s_\alpha}x-\frac{1}{2}(T_b y+y)\|\\ &\leq\sup\{\|T_{ba_0}T_{s_\alpha}x-\frac{1}{2}(T_b y+y)\| : s_\alpha\geq s_\alpha\}\\ &<r,\end{aligned}$$

which contradicts $r=\inf_{s\in S} r_s$ and the proof is complete.

Similarly, using Lemma 2.1 and Lemma 2.2, we get

THEOREM 2.4. *Let X be reflexive and satisfy Opial's condition and $x\in C$. Then, the orbit $\{T_s x : s\in S\}$ converges weakly to an element of $F(\mathfrak{I})$ if and only if $E(x)\neq\phi$ and $T_{ts}x-T_s x\to\phi$* for all $t\in S$.

Proof. Since $E(x)\neq\phi$, the orbit $\Im(x)=\{T_s x : s\in S\}$ is bounded. By reflexivity of $X, \omega_w(x)\neq\phi$. By Lemma 2.1 and Lemma 2.2, it suffices to show that $\omega_w(x)\subseteq F(\Im)$. Let $y\in\omega_w(x)$; hence there is a subnet $\{T_{s_\alpha}x\}$ of the net $\{T_s x : s\in S\}$ such that $T_{s_\alpha}x\rightharpoonup y$. Given $\varepsilon>0$, there is $t_0\in S$ such that $k_t\leq 1+\varepsilon/M$ for all $t\geq t_0$, where $M=\sup_{s\in S}\|T_s x-y\|<\infty$. Then, for each α and for all $t\geq t_0$, we have

$$\begin{aligned}\|T_tT_{s_\alpha}x-T_ty\| &\leq k_t\|T_{s_\alpha}x-y\|\\ &\leq\left(1+\frac{\varepsilon}{M}\right)\|T_{s_\alpha}x-y\|\\ &\leq\|T_{s_\alpha}x-y\|+\varepsilon.\end{aligned}$$

So,

$$\begin{aligned}\|T_{s_\alpha}x-T_ty\| &\leq\|T_{s_\alpha}x-T_{ts_\alpha}x\|+\|T_tT_{s_\alpha}x-T_ty\|\\ &\leq\|T_{s_\alpha}x-T_{ts_\alpha}x+\|T_{s_\alpha}x-y\|+\varepsilon.\end{aligned}$$

Then, since $T_{ts_\alpha}x-T_{s_\alpha}x\to 0$ for all $t\in S$, we have

$$\limsup_\alpha\|T_{s_\alpha}x-T_ty\|\leq\limsup_\alpha\|T_{s_\alpha}x-y\|+\varepsilon.$$

Since ε is arbitrary, it follows from Opial's condition(*) that $T_ty=y$ for all $t\geq t_0$, and so $y\in F(\Im)$.

Throughout the slight modification of the proof of Theorem 2.2 in [5], we can obtain the following:

THEOREM 2.5. *Let X be unifomly convex and $\Im=\{T_s : s\in S\}$ left reversible. If there is a point $x\in C$ such that its orbit $\Im(x)=\{T_s x : s\in S\}$ is bounded, then the asymptotic center $c(x)$ of $\Im(x)$ with respect to C is in fact a common fixed point of $\Im$.*

As immediate consequence, joining with Theorem 2.3, we get the following;

COROLLARY 2.6. *Let X be uniformly convex and satisfy Opial's condition and $x\in C$. If the semigroup $\Im=\{T_s : s\in S\}$ is reversible, and if the orbit $\Im(x)$ of x is bounded, then the orbit $\{T_s x : s\in S\}$ converges weakly to the asymptotic center $c(x)$ of $\Im(x)$ with respect to C if $T_{ts}x-T_s x\to 0$ for all $t\in S$.*

Proof. By Theorem 2.3, let $y\in F(\Im)$ be such that $T_s x\rightharpoonup y$. Since $c(x)\in F(\Im)$ in Theorem 2.5, it follows from Opial's condition (*) that

$$\lim_s \sup \| T_s x - y \| \leq \lim_s \sup \| T_s x - c(x) \|.$$

Then, the uniqueness of asymptotic center implies that $y = c(x) \in F(\mathfrak{J})$ (see[2]).

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