DERIVATIONS ON COMMUTATIVE BANACH ALGEBRAS

YOUNG-WHAN LEE AND KIL-WOUNG JUN

1. Introduction

If $T$ is a linear operator from a Banach space $X$ into a Banach space $Y$, then the separating space $\mathcal{E}(T)$ of $T$ is defined by $\mathcal{E}(T) = \{ y \in Y \mid \text{there is } x_n \to 0 \text{ in } X \text{ with } Tx_n \to y \text{ in } Y \}$. Note that $\mathcal{E}(T) = \{0\}$ if and only if $T$ is continuous, by Closed Graph Theorem. A derivation on a Banach algebra $A$ is a linear mapping $D$ of $A$ into itself such that $D(ab) = a(Db) + (Da)b$ ($a, b \in A$).

In [6] Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the radical of $A$ and conjectured that the assumption of continuity is unnecessary. In [2] Cusack showed that if any derivation $D$ on commutative Banach algebra $A$ has a nilpotent separating space, then the range of $D$ is contained in the radical of $A$. But it is an open question that every derivation on a commutative Banach algebra has a nilpotent separating space.

In this paper we show that if there is a derivation on a commutative Banach algebra which has a non-nilpotent separating space, then there is a discontinuous derivation on a commutative Banach algebra which has a range in its radical. Also we show that if every prime ideal is closed in a commutative Banach algebra with identity then every derivation on it has a range in its radical.

2. Derivations on commutative Banach algebras

In [4] Khosravi showed that $D$ is a derivation on a commutative Banach algebra $A$ such that $\mathcal{E}(D^n) \subseteq R$ for all $n \geq 1$, then $DA \subseteq R$ where $R$ is the radical of $A$. Note that $\mathcal{E}(D^n) \subseteq R$ if and only if $\phi \circ D^n$ is continuous for any multiplicative linear functional $\phi$ on $A$.

Received March 26, 1988.
We need the following lemma to prove our main results.

**Lemma 1.** Let $A$ be a commutative Banach algebra with the radical $R$. If $D : A \to A$ is a derivation such that $\frac{x \mathcal{E}(D)}{x} = \mathcal{E}(D)$ for all nonzero $x \in A$, then $DA \subseteq R$.

**Proof.** By hypothesis, $\mathcal{E}(D) \subseteq R$ and so $\phi \circ D$ is continuous for all $\phi \in \mathcal{E}_A$, where $\mathcal{E}_A$ is the set of all multiplicative linear functionals on $A$. Suppose that $\phi \in \mathcal{E}_A$ and $y \in \mathcal{E}(D(\mathcal{E}(D)))$. Let $x \in \mathcal{E}(D)$ with $y = \phi \circ D(x)$. Then there is a sequence $\{y_n\}$ in $\mathcal{E}(D)$ such that $x = \lim_{n \to \infty} xy_n$.

Then

$$y = \lim_{n \to \infty} \phi(x(Dy_n)) + y_n(Dx)$$
$$= \lim_{n \to \infty} [\phi(x)\phi(Dy_n) + \phi(y_n)\phi(Dx)]$$
$$= 0.$$

Thus $\phi \circ D(\mathcal{E}(D)) = \{0\}$ for all $\phi \in \mathcal{E}_A$. Since $\frac{\phi \circ D(\mathcal{E}(D))}{\phi \circ D(\mathcal{E}(D)^2)} = \mathcal{E}(\phi \circ D^2) = \{0\}$ for all $\phi \in \mathcal{E}_A$, $\mathcal{E}(D^2) \subseteq R$. Suppose that $\mathcal{E}(D^i) \subseteq R$ for each $i \leq m$ and $y \in \mathcal{E}(D^m(\mathcal{E}(D)))$ for $\phi \in \mathcal{E}_A$. Then there are $x \in \mathcal{E}(D)$ and a sequence $\{y_n\}$ in $\mathcal{E}(D)$ such that $y = \phi \circ D^m(x)$ and $x = \lim_{n \to \infty} xy_n$. Therefore

$$y = \lim_{n \to \infty} \phi \circ D^m(xy_n)$$
$$= \lim_{n \to \infty} \phi \left[ \sum (D^{n-1}x)(D^i y_n) \right] \quad \text{(by Leibnitz rule)}$$
$$= \lim_{n \to \infty} \left[ \sum (D^{n-1}x)\phi(D^iy_n) \right]$$
$$= 0$$

since $\phi \circ D^i(\mathcal{E}(D)) = \{0\}$ for $i < m$.

Thus $\phi \circ D^m(\mathcal{E}(D)) = \{0\}$ for all $\phi \in \mathcal{E}_A$. Since $\phi \circ D^m(\mathcal{E}(D)) = \mathcal{E}(\phi \circ D^{m+1}) = \mathcal{E}(\mathcal{E}(D^{m+1})) = \{0\}$, $\mathcal{E}(D^{m+1}) \subseteq R$. By induction, $\mathcal{E}(D^n) \subseteq R$ for all $m \geq 1$. By Khosravi Theorem, we have $D(A) \subseteq R$.

Note that if $D$ is a derivation on a commutative Banach algebra $A$ and $K_D(I) = \{x \in I : D^n x \in I \text{ for all } n \geq 1\}$ where $I$ is an ideal of $A$, then $K_D(I)$ is an ideal, and if $I$ is a prime ideal then $K_D(I)$ is a prime ideal [3].

**Remark.** By “Prime Ideal Theorem” in [1], we know that if $D$ is a discontinuous derivation from a commutative Banach algebra
Derivations on commutative Banach algebras

A into itself, then there is a discontinuous derivation \( D_0=a_0D \) for some \( a_0\in A \) such that

1. for each \( a\in A \), either \( a\overline{\mathcal{E}(D_0)}=\mathcal{E}(D_0) \) or \( a\mathcal{E}(D_0) = \{0\} \)
2. \( I_0=\{a\in A|a\mathcal{E}(D_0)=\{0\}\} \) is a prime ideal in \( A \).

**Theorem 1.** If there is a derivation on a commutative Banach algebra which has a non-nilpotent separating space, then there is a discontinuous derivation on a commutative Banach algebra which has a range in its radical.

**Proof.** Suppose that there is a derivation \( D \) on a commutative Banach algebra \( A \) such that \( \mathcal{E}(D) \) is non-nilpotent. We may assume that \( A \) has an identity. Since \( \mathcal{E}(D) \) is a separating ideal there is a minimal prime ideal \( P \) such that \( \mathcal{E}(D)\subseteq P \) and \( P \) is closed [2]. Then \( A/P \) is a commutative prime Banach algebra. By the minimality of \( P \), \( K_0(P)=P \) and so \( D(P)\subseteq P \). Thus we can define a derivation \( \overline{D} \) on \( A/P \) by \( \overline{D}(a+P)=Da+P \). By Lemma 1.4 in [5], \( \overline{D} \) is discontinuous. By Prime Ideal Theorem there is a discontinuous derivation \( D_0=(a_0+P)\overline{D} \) for some \( a_0\in A/P \) such that for each \( a\in A \), either \( (a+P)\mathcal{E}(D_0)=\mathcal{E}(D_0) \) or \( (a+P)\mathcal{E}(D_0)=P \). But \( (a+P)\mathcal{E}(D_0)=P \) does not happen for \( a\in A/P \) because \( A/P \) is an integral domain. Therefore for any nonzero \( a+P \) in \( A/P \), \( (a+P)\mathcal{E}(D_0)=\mathcal{E}(D_0) \). By Lemma 1, we complete the proof.

Garimella [3] showed that if \( A \) is a commutative semi-prime Banach algebra with identity such that every prime ideal is closed, then every derivation on \( A \) is continuous. From this we get the following result.

**Theorem 2.** Every derivation on a commutative Banach algebra \( A \) with identity in which every prime ideal is closed has a range in its radical.

**Proof.** Let \( L \) be a prime radical of \( A \). Note that \( L \) is the intersection of all prime ideals and it contains of all the nilpotent elements. Then \( L \) is closed and \( A/L \) is a commutative semi-prime Banach algebra. Note that \( D(L)\subseteq L \) [2, Lemma 4.1], and it is easy to see that every prime ideal in \( A/L \) is closed. Hence we can
define a derivation $D$ on $A/L$ by $D(a+L)=Da+L$. Then $D$ is continuous by Garimella Theorem and so $\mathfrak{E}(D)\subseteq L$ [5, Lemma 1.4]. Therefore $\mathfrak{E}(D)$ is nilpotent and Cusack Theorem implies that $DA\subseteq R$, where $R$ is the radical of $A$.

References


University of Taejon
Taejon 302-120, Korea
and
Chungnam National University
Taejon 302-764, Korea