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DERIVATIONS ON COMMUTATIVE BANACH ALGEBRAS

YOUNG-WHAN LEE AND KIL-WOUNG JUN

1. Introduction

If T is a linear operator from a Banach space X into a Banach space Y, then the separating space $\mathfrak{S}(T)$ of T is defined by $\mathfrak{S}(T)=\{y\in Y|$there is $x_n\to 0$ in X with $Tx_n\to y$ in $Y\}$. Note that $\mathfrak{S}(T)=\{0\}$ if and only if T is continuous, by Closed Graph Theorem. A derivation on a Banach algebra A is a linear mapping D of A into itself such that $Dab=a(Db)+(Da)b$ $(a,b\in A)$.

In [6] Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the radical of A and conjectured that the assumption of continuity is unnecessary. In [2] Cusack showed that if any derivation D on commutative Banach algebra A has a nilpotent separating space, then the range of D is contained in the radical of A. But it is an open question that every derivation on a commutative Banach algebra has a nilpotent separating space.

In this paper we show that if there is a derivation on a commutative Banach algebra which has a non-nilpotent separating space, then there is a discontinuous derivation on a commutative Banach algebra which has a range in its radical. Also we show that if every prime ideal is closed in a commutative Banach algebra with identity then every derivation on it has a range in its radical.

2. Derivations on commutative Banach algebras

In [4] Khosravi showed that D is a derivation on a commutative Banach algebra A such that $\mathfrak{S}(D^n)\subseteq R$ for all $n\geq 1$, then $DA\subseteq R$ where R is the radical of A. Note that $\mathfrak{S}(D^n)\subseteq R$ if and only if $\phi\circ D^n$ is continuous for any multiplicative linear functional ϕ on A.

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We need the following lemma to prove our main results.

LEMMA 1. *Let A be a commutatve Banach algebra with the radical R. If $D : A \to A$ is a derivation such that $\overline{x\mathfrak{S}(D)} = \mathfrak{S}(D)$ for all nonzero $x \in A$, then $DA \subseteq R$.*

Proof. By hypothesis, $\mathfrak{S}(D) \subseteq R$ and so $\phi \circ D$ is continuous for all $\phi \in \Phi_A$, where Φ_A is the set of all multiplicative linear functionals on A. Suppose that $\phi \in \Phi_A$ and $y \in \phi \circ D(\mathfrak{S}(D))$. Let $x \in \mathfrak{S}(D)$ with $y = \phi \circ D(x)$. Then there is a sequence $\{y_n\}$ in $\mathfrak{S}(D)$ such that $x = \lim_{n\to\infty} xy_n$.

Then

$$\begin{aligned} y &= \lim_{n\to\infty} \phi(x(Dy_n)) + y_n(Dx)) \\ &= \lim_{n\to\infty} [\phi(x)\phi(Dy_n) + \phi(y_n)\phi(Dx)] \\ &= 0. \end{aligned}$$

Thus $\phi \circ D(\mathfrak{S}(D)) = \{0\}$ for all $\phi \in \Phi_A$. Since $\overline{\phi \circ D\mathfrak{S}(D)} = \mathfrak{S}(\phi \circ D^2) = \overline{\phi(\mathfrak{S}(D^2))} = \{0\}$ for all $\phi \in \Phi_A$, $\mathfrak{S}(D^2) \subseteq R$. Suppose that $\mathfrak{S}(D^i) \subseteq R$ for each $i \leq m$ and $y \in \phi \circ D^m(\mathfrak{S}(D))$ for $\phi \in \Phi_A$. Then there are $x \in \mathfrak{S}(D)$ and a sequence $\{y_n\}$ in $\mathfrak{S}(D)$ such that $y = \phi \circ D^m(x)$ and $x = \lim_{n\to\infty} xy_n$. Therefore

$$\begin{aligned} y &= \lim_{n\to\infty} \phi \circ D^m(xy_n) \\ &= \lim_{n\to\infty} \phi[\textstyle\sum \binom{m}{i} (D^{m-i}x)(D^i y_n)] \quad \text{(by Leibnitz rule)} \\ &= \lim_{n\to\infty} [\textstyle\sum \binom{m}{i} \phi(D^{m-i}x)\phi(D^i y_n)] \\ &= 0 \end{aligned}$$

since $\phi \circ D^i(\mathfrak{S}(D)) = \{0\}$ for $i < m$.

Thus $\phi \circ D^m(\mathfrak{S}(D)) = \{0\}$ for all $\phi \in \Phi_A$. Since $\overline{\phi \circ D^m\mathfrak{S}(D)} = \mathfrak{S}(\phi \circ D^{m+1}) = \overline{\phi(\mathfrak{S}(D^{m+1}))} = \{0\}$, $\mathfrak{S}(D^{m+1}) \subseteq R$. By induction, $\mathfrak{S}(D^m) \subseteq R$ for all $m \geq 1$. By Khosravi Theorem, we have $D(A) \subseteq R$.

Note that if D is a derivation on a commutative Banach algebra A and $K_D(I) = \{x \in I : D^n x \in I \text{ for all } n \geq 1\}$ where I is an ideal of A, then $K_D(I)$ is an ideal, and if I is a prime ideal then $K_D(I)$ is a prime ideal [3].

REMARK. By "Prime Ideal Theorem" in [1], we know that if D is a discontinuous derivation from a commutative Banach algebra

A into itself, then there is a discontinuous derivation $D_0 = a_0 D$ for some $a_0 \in A$ such that

(1) for each $a \in A$, either $\overline{a\mathfrak{S}(D_0)} = \mathfrak{S}(D_0)$ or $a\mathfrak{S}(D_0) = \{0\}$

(2) $I_0 = \{a \in A \mid a\mathfrak{S}(D_0) = \{0\}\}$ is a prime ideal in A.

THEOREM 1. *If there is a derivation on a commutative Banach algebra which has a non-nilpotent separating space, then there is a discontinuous derivation on a commutative Banach algebra which has a range in its radical.*

Proof. Suppose that there is a derivation D on a commutative Banach algebra A such that $\mathfrak{S}(D)$ is non-nilpotent. We may assume that A has an identity. Since $\mathfrak{S}(D)$ is a separating ideal there is a minimal prime ideal P such that $\mathfrak{S}(D) \not\subset P$ and P is closed [2]. Then A/P is a commutative prime Banach algebra. By the minimality of P, $K_D(P) = P$ and so $D(P) \subseteq P$. Thus we can define a derivation $\bar{D}$ on A/P by $\bar{D}(a+P) = Da + P$. By Lemma 1.4 in [5], $\bar{D}$ is discontinuos. By Prime Ideal Theorem there is a discontinuous derivation $D_0 = (a_0 + P)\bar{D}$ for some $a_0 \in A/P$ such that for each $a \in A$, either $\overline{(a+P)\mathfrak{S}(D_0)} = \mathfrak{S}(D_0)$ or $(a+P)\mathfrak{S}(D_0) = P$. But $(a+P)\mathfrak{S}(D_0) = P$ does not happen for $a \in A/P$ because A/P is an integral domain. Therefore for any nonzero $a+P$ in A/P, $\overline{(a+P)\mathfrak{S}(D_0)} = \mathfrak{S}(D_0)$. By Lemma 1, we complete the proof.

Garimella [3] showed that if A is a commutative semi-prime Banach algebra with identity such that every prime ideal is closed, then every derivation on A is continuous. From this we get the following result.

THFOREM 2. *Every derivation on a commutative Banach algebra A with identity in which every prime ideal is closed has a range in its radical.*

Proof. Let L be a prime radical of A. Note that L is the intersection of all prime ideals and it contains of all the nilpotent elements. Then L is closed and A/L is a commutative semi-prime Banach algebra. Note that $D(L) \subseteq L$ [2, Lemma 4.1], and it is easy to see that every prime ideal in A/L is closed. Hence we can define a derivation D on A/L by $D(a+L)=Da+L$. Then D is continuous by Garimella Theorem and so $\mathfrak{S}(D)\subseteq L$ [5, Lemma 1.4]. Therefore $\mathfrak{S}(D)$ is nilpotent and Cusack Theorem implies that $DA\subseteq R$, where R is the radical of A.

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University of Taejon
Taejon 302-120, Korea
and
Chungnam National University
Taejon 302-764, Korea