SOME PERMANENTAL INEQUALITIES

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1. Introduction

Let $Q_n$ and $Pmf_n$ denote the sets of all $n \times n$ doubly stochastic matrices and the set of all $n \times n$ permutation matrices respectively. For $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$ we write $A \leq B$ ($A < B$) to mean that $a_{ij} \leq b_{ij}$ ($a_{ij} < b_{ij}$) for all $i = 1, \ldots, m$; $j = 1, \ldots, n$. Let $I_n$ denote the identity matrix of order $n$, let $J_n$ denote the $n \times n$ matrix all of whose entries are $1/n$, and let $K_n = nJ_n$. For a complex square matrix $A$, the permanent of $A$ is denoted by per $A$. Let $E_{ij}$ denote the matrix of suitable size all of whose entries are zeros except for the $(i, j)$-entry which is one.

For an $n \times n$ matrix $A$ and for $i_1, \ldots, i_s$, $j_1, \ldots, j_s \subseteq \{1, \ldots, n\}$, let $A(i_1, \ldots, i_s, j_1, \ldots, j_s)$ denote the matrix obtained from $A$ by deleting the rows $i_1, \ldots, i_s$ and the columns $j_1, \ldots, j_s$.

For positive integral $n$-vectors $R, S$ and nonnegative $n \times n$ matrices $A, B$, let $U_{R,S}(A, B)$ denote the set of all $n \times n$ matrices $X$ whose row sum vector and column sum vector are $R$ and $S$ respectively and such that $A \leq X \leq B$.

The sets $U_{R,S}(A, B)$ with $A \leq B \leq K_n$ have been studied in [1] as faces of the so called assignment polytope $U_{R,S}(O, K_n)$.

In general, it is very hard to determine the minimum and the maximum values of permanent function on the set $U_{R,S}(A, B)$ even with some good restrictions on the vectors $R$ and $S$ as well as on the bound matrices $A$ and $B$.

A particular case of $U_{R,S}(A, B)$ with $R = S = (1, \ldots, 1)$, $A = O$, $B = K_n$ is the set $Q_n$, on which the minimum permanent is achieved uniquely at $J_n$. This result was conjectured in 1926 by van der Waerden and proved by Egorycev in 1980, and now is called the van der Waerden-Egorycev's theorem in the literature.

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Marcus and Minc [6] conjectured that, for any $A \in \mathcal{O}_n$, $n \geq 2$, \[
\per\left(\frac{n J_n - A}{n - 1}\right) \leq \per A
\]
with equality if and only if $A = J_n$ for $n \geq 4$, and proved their conjecture for positive semi-definite symmetric doubly stochastic matrices.

In [7], E.T.H. Wang proved Marcus-Minc' conjecture for the case of $n = 3$ and proposed a conjecture asserting that \[
\per\left(\frac{n J_n + A}{n + 1}\right) \leq \per A
\]
for all $A \in \mathcal{O}_n$, $n \geq 2$.

Marcus-Minc' conjecture was proved to be true for the case of $n = 4$ by T. Foregger [4]. Both of these two conjectures are true for doubly stochastic matrices in a sufficiently small neighbourhood of $J_n$ because of the van der Waerden-Egorycev's theorem. Recently, D.K. Chang [2], [3] has proven the validity of these two conjectures in the complement of a sufficiently large neighbourhood of $J_n$ by showing first that \[
(1) \quad \per\left(\frac{n J_n - A}{n - 1}\right) \leq \frac{d_n}{(n - 1)^n}
\]
where $d_n$ denotes the $n$-th derangement number $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$, and \[
(2) \quad \per\left(\frac{n J_n + A}{n + 1}\right) \leq \per\left(\frac{n J_n + I_n}{n + 1}\right)
\]
for all $A \in \mathcal{O}_n$, $n \geq 2$.

Let $E_n = (1, \ldots, 1)$, the $n$-tuple of ones. In this paper we determine the set of permanent-maximal matrices in $(U_{k,s}(A, B)$ along with the maximum value of the permanent function for the case of $R = S = (n - 1)E_n$, $A = O$, $B = K_n$ or of $R = S = (n + 1)E_n$, $A = K_n$, $B = 2K_n$, by a simple combinatorial argument. And as a corollary, we will have the following permanental inequality; \[
(3) \quad \per\left(\frac{n J_n + A}{n + 1}\right) \leq \frac{n!}{(n + 1)^n} \sum_{k=0}^{n} \frac{(-1)^{\frac{k}{2}}}{k!}
\]
for all $A \in \mathcal{O}_n$, $n \geq 2$, where the signs $+$, $-$ occur in the same
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order.

Notice that the inequality (3) is the combination of the inequalities (1) and (2). In addition, we also show that equality in (3) holds if and only if $A \in \text{Pmt}_n$, for $n \geq 4$.

Finally, we find out some subclass of $\mathcal{Q}_n$ over which the two conjecture are valid.

2. Maximum permanent on $U_{R,S}(A, B)$

In this section, we obtain the maximum value of permanent on $U_{R,S}(A, B)$ and find out the matrices in $(U_{R,S}(A, B))$ at which the maximum value is achieved for the case of $R = S = (n-1)E_n$, $A = O$, $B = K_n$ or of $R = S = (n+1)E_n$, $A = K_n$, $B = 2K_n$.

For this purpose only, let

$U_{n-1} = \{X = [x_{ij}] | \sum_{j=1}^{n} x_{ij} = n-1 \ (i = 1, \cdots, n), \ O \leq X \leq K_n\},$

$U_{n+1} = \{X = [x_{ij}] | \sum_{j=1}^{n} x_{ij} = n+1 \ (i = 1, \cdots, n), \ K_n \leq X \leq 2K_n\}.$

Lemma 1. The permanent function attains its maximum on each of $U_{n-1}$ and $U_{n+1}$ at an integral matrix.

Proof. Let $A \in U_{n-1}$ be such that $\text{per } A \geq \text{per } X$ for all $X \in U_{n-1}$ with as few non-integral entries as possible.

Suppose that $A$ is not an integral matrix. Then there are $i, j$ such that $0 < a_{ij} < 1$. Since the $i$-th row sum of $A$ is an integer, there is some $l$, $l \neq j$, such that $0 < a_{il} < 1$. For a real number $\varepsilon$ with sufficiently small absolute value, let $A_\varepsilon = A + \varepsilon (E_{ij} - E_{il})$. Then $A_\varepsilon \in U_{n-1}$ and

$$\text{per } A_\varepsilon = \text{per } A + \varepsilon (\text{per } A(i|j) - \text{per } A(i|l)).$$

Hence it follows that $\text{per } A(i|j) = \text{per } A(i|l)$. Therefore $f(\varepsilon) = \text{per } A_\varepsilon$ is a constant function of $\varepsilon$. Now, by choosing a suitable $\varepsilon$, we come to get a permanent-maximal matrix $A_\varepsilon \in U_{n-1}$ with strictly fewer non-integral entries than $A$, contradicting the choice of $A$.

Thus the assertion for $U_{n-1}$ is proved.

Similarly, we can prove the lemma for $U_{n+1}$.

Lemma 2. Let $U_{n-1}^*, U_{n+1}^*$ denote the sets of all integral matrices in $U_{n-1}$, $U_{n+1}$ respectively. Then
(1) For all $A \in U_{n-1}^*$, per $A \leq n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$, with equality if and only if $A = K_n - P$ for some $P \in \text{Pmt}_n$, for $n \geq 4$.

(2) For all $A \in U_{n+1}^*$, per $A \leq n! \sum_{k=0}^{n} \frac{1}{k!}$, with equality if and only if $A = K_n + P$ for some $P \in \text{Pmt}_n$.

Proof. (1) Let $U = [u_{ij}] \in U_{n+1}^*$ be such that per $X \leq \text{per} \, U$ for all $X \in U_{n+1}^*$. Notice that every row of $U$ is an $n$-vector all of whose components are 1's except for exactly one which is 0. Let $C = (c_1, \ldots, c_n)$ be the column sum vector of $U$. Without loss of generality, we may assume that $c_1 = \max \{c_1, \ldots, c_n\}$ and $c_2 = \min \{c_1, \ldots, c_n\}$. Then $c_1 \geq n-1$ and $c_2 \leq n-1$.

Suppose that $c_1 > n-1$. Then $c_1 = n$ so that $u_{11} = \cdots = u_{n1} = 1$. Since, in this case, $c_2 \leq n-2$, we may also assume that $u_{12} = u_{22} = 0$. Then it follows that per $U(1|1) \leq \text{per} \, U(1|2)$. Moreover if $n \geq 4$, per $U (1|1) < \text{per} \, U(1|2)$. For, if $n \geq 5$, then per $U(1,2|1,2) > 0$ by Frobenius-König's theorem. If $n = 4$, per $U(1,2|1,2) = 0$ only if $U(1,2|1,2)$ has a zero column so that $U$ must be a permutation equivalent to

$$
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
\end{bmatrix}
$$

But, if it is the case, per $U = 8 < 9 = \text{per}(K_3 - I_3)$, which is impossible because of the choice of $U$ and because $K_3 - I_3 \in U_{n-1}$.

Let $H = U - E_{11} + E_{12}$. Then $H \in U_{n+1}^*$ and per $H \geq \text{per} \, U$. If $n \geq 4$, we have per $H > \text{per} \, U$, contradicting the maximality of per $U$. Therefore it must be that $c_1 = n-1$ so that $c = (n-1)E_n$, in which case it must be that $U = K_n - P$ for some $P \in \text{Pmt}_n$.

In the case $n = 3$, if the column sum vector of $H$ is different from $2E_3$, then we do the same job as above to get a matrix $M \in U_{n+1}^*$ with row sum vector $2E_3$ and per $M \geq \text{per} \, H$. But then $M = K_3 - P$ for some $P \in \text{Pmt}_3$ and per $M = \text{per} \, U$ by the maximality of per $U$. Now since per $(K_n - P) = \text{per}(K_n - I_n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ for all $P \in \text{Pmt}_n$, the proof is completed.

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(2) Again let $U \in U_{n+1}^*$ be such that per $X \leq$ per $U$ for all $X \in U_{n+1}^*$ and let $C = (c_1, \ldots, c_n)$ be the column sum vector of $U$. Notice, in this case, that every row of $U$ is an $n$-vector, of the components of which $n-1$ are 1's and one is 2. Since $U$ is a positive matrix, every submatrix of $U$ has positive permanent. With these facts in mind, we can show that $C = (n+1)E_n$ regardless of whether $n > 3$ or not by a similar argument as the one used in the proof of (1). But then $U = K_n + P$ for some $P \in \text{Pmt}_n$. Now the assertion (2) follows because per $(K_n + P) = \text{per}(K_n + I_n) = n! \sum_{k=0}^{n} (1/k!)$ for all $P \in \text{Pmt}_n$.

Since, for every $P \in \text{Pmt}_n$, $K_n - P \in U_{(n-1)E_n, (n-1)E_n} (0, K_n)$ and $K_n + P \in U_{(n+1)E_n, (n+1)E_n} (K_n, 2K_n)$, the combination of lemmas 1 and 2 tells us that, for every $A \in U_{(n-1)E_n, (n-1)E_n} (0, K_n)$, per $A \leq \text{per}(K_n - P)$, and for every $A \in U_{(n+1)E_n, (n+1)E_n} (K_n, 2K_n)$, per $A \leq \text{per}(K_n + P)$. Thus we have proven half of the following

**Theorem 3.** (1) For any $A \in U_{(n-1)E_n, (n-1)E_n} (0, K_n)$,

$$\text{per } A \leq n! \sum_{k=1}^{n} \frac{(-1)^k}{k!}$$

with equality if and only if $A = K_n - P$ for some $P \in \text{Pmt}_n$, if $n \geq 4$.

(2) For any $A \in U_{(n+1)E_n, (n+1)E_n} (K_n, 2K_n)$,

$$\text{per } A \leq n! \sum_{k=0}^{n} \frac{1}{k!}$$

with equality if and only if $A = K_n + P$ for some $P \in \text{Pmt}_n$.

**Proof.** We need only to show that, for $n \geq 4$, every permanent-maximal matrix on $U_{n-1}$ is a $(0,1)$-matrix. Suppose that there is a non $(0,1)$ permanent-maximal matrix $A = [a_{ij}]$ in $U_{n-1}$, then we can pick up such an $A$ with exactly two non integral entries which are in a same row, say $a_{11}$ and $a_{12}$. Then, by Lemma 2, we may assume that

$$A = \begin{bmatrix} a_{11} & a_{12} & 1 & \cdots & 1 \\ 1 & \cdots & K_{n-1} - I_{n-1} \\ \vdots & \ddots & 1 \\ 1 & \cdots & 1 \end{bmatrix}$$
showing us that per $A(1|1)<per A(1|2)$ and hence that for $\varepsilon$, $0<\varepsilon<\min\{a_{11},a_{12}\}$, per$(A+\varepsilon(E_{12}-E_{11})))>per A$ even if $A+\varepsilon(E_{12}-E_{11})\in U_{n-1}$, a contradiction. Therefore there can not be such an $A$ and the proof is completed.

Since 
\[
\left\{ \frac{1}{n-1}A \mid A \in U_{(n-1)E_n,(n-1)E_n}(O, K_n) \right\} = \left\{ \frac{K_n-S}{n-1} \mid S \in \Omega_n \right\}
\]
and
\[
\left\{ \frac{1}{n+1}A \mid A \in U_{(n+1)E_n,(n+1)E_n}(K_n, 2K_n) \right\} = \left\{ \frac{K_n+S}{n+1} \mid S \in \Omega_n \right\},
\]
We have the following

**Corollary.** For any $S \in \Omega_n$,

\[
\text{per} \left( \frac{nJ_n-S}{n-1} \right) \leq \frac{n!}{(n-1)^n} \sum_{k=0}^{n} \frac{(-1)^k}{k!}
\]

with equality if and only if $S \in \text{Pmt}_n$ if $n \geq 4$, and

\[
\text{per} \left( \frac{nJ_n+S}{n+1} \right) \leq \frac{n!}{(n+1)^n} \sum_{k=0}^{n} \frac{1}{k!}
\]

with equality if and only if $S \in \text{Pmt}_n$.

3. **Permanents of partly decomposable matrices**

In this section, we are to show that every partly decomposable doubly stochastic matrix satisfies the Marcus-Minc conjecture and Wang's conjecture.

An $n \times n$ matrix is called *partly decomposable* if it contains an $s \times t$ zero submatrix with $s+t=n$.

Let $D$ be an $n \times n$ $(0,1)$-matrix with per $D \neq 0$, then $\Omega(D) = \{X \in \Omega_n \mid X \leq D\}$ is a face of $\Omega_n$ [5].

Let $p,q$ be positive integers such that $p+q=n$. Suppose that $D = \begin{bmatrix} K_p & 0 \\ * & K_q \end{bmatrix}$ is a $(0,1)$-matrix. It is known that $J_p \oplus J_q$ is the unique permanent-minimal matrix on $\Omega(D)$. (see [5], for example).

From now on, in the sequel, let $\delta^k = k! / k^k$ for $k = 1, 2, \ldots$.

**Theorem 4.** Let $A$ be any partly decomposable matrix in $\Omega_n$, then per $A \geq \delta_{n-1}$ with equality if and only if $A = P(I_1 \oplus J_{n-1})Q$ for some $P,Q \in \text{Pmt}_n$. 

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Proof. Let \( A \in \Omega_n \) be partly decomposable such that \( \text{per} \ A \leq \text{per} \ X \) for all partly decomposable \( X \in \Omega_n \). Then \( A = J_p \oplus J_q \) for some integers \( p, q \) with \( p, q \geq 1 \) and \( p + q = n \), so that \( \text{per} \ A = \delta_p \delta_q \). We are to show that either \( p = 1 \) or \( q = 1 \). Suppose that \( q \geq p > 1 \). Then

\[
\delta_{p-1} \delta_{q+1} = \left( \frac{p}{p-1} \right)^{q-1} \left( \frac{q}{q+1} \right)^q \delta_p \delta_q < \delta_p \delta_q
\]

showing us that \( \text{per} (J_{p-1} \oplus J_{q+1}) < \text{per} (J_p \oplus J_q) \). Therefore it must be that \( p = 1 \) by the minimality of \( J_p \oplus J_q \), and we are done.

To prove the validity of Marcus-Minc conjecture and Wang's conjecture for partly decomposable doubly stochastic matrices, it suffices to show only that

\[
\frac{n!}{(n+1)^n} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \leq \delta_{n-1}
\]

where the signs +, − are written in the same order. But since

\[
\sum_{k=0}^{n} \frac{(-1)^k}{k!} < \sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k}{k!} = \frac{3}{8} \quad \text{and} \quad \frac{3}{8} \frac{n}{n-1} < 1 \quad \text{for} \quad n \geq 4,
\]

we have

\[
\frac{n!}{(n-1)^n} \sum_{k=0}^{n} \frac{(-1)^k}{k!} < \frac{3}{8} \frac{n}{n-1} \delta_{n-1} < \delta_{n-1}.
\]

On the other hand, we have, for all \( n > 2 \),

\[
\left( \frac{n+1}{n} \right)^n \left( \frac{n}{n-1} \right)^{-n-1} \geq \left( \frac{3}{2} \right)^2 \frac{2}{1} = 4.5 > e
\]

that is

\[
\left( \frac{n}{n+1} \right)^n e < \left( \frac{n}{n-1} \right)^{-n-1}.
\]

Thus we have

\[
\frac{n!}{(n+1)^n} \sum_{k=0}^{n} \frac{1}{k!} < \left( \frac{n}{n+1} \right)^n e \delta_n < \delta_{n-1}
\]

since \( \delta_{n-1} = \left( \frac{n}{n-1} \right)^{-n-1} \delta_n \).

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Since Marcus-Minc conjecture is already known to be true for \( n=3 \), by the above discussions we have the following

**Theorem 5.** For any partly decomposable \( S \in \Omega_n \), \( n \geq 2 \),

\[
\text{per}\left( \frac{nI_n+S}{n \pm 1} \right) \leq \text{per} \ S
\]

where the signs \( +, - \) are written in the same order.

References


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