HYPOELLIPTICITY OF SYSTEMS OF
ANALYTIC VECTOR FIELDS

K. H. KWON AND B. C. SONG

1. Introduction

In this paper, we are concerned with the pointwise-hypoellipticity (see Definition 2.1) of an $m$-dimensional Frobenius Lie algebra $L$ of analytic complex vector fields in some open subset $\Omega$ of $\mathbb{R}^{m+1}$. That is, $L$ is a set of complex vector fields in $\Omega$ with (real-) analytic coefficients satisfying:

(A) each point of $\Omega$ has an open neighborhood in which $L$ is generated by $m$ linearly independent elements of $L$;

(B) $L$ is closed under the commutation bracket $[A, B]$.

The pointwise-analytic hypoellipticity of $L$ is completely characterized by M. S. Baouendi and F. Treves in [1]. Here, we shall prove that if $L$ is hypoelliptic at a point then it must be analytic hypoelliptic in a full neighborhood of the same point. When the coefficients are $C^\infty$, hypoellipticity of $L$ was discussed in [2].

By shrinking $\Omega$ about a point in $\Omega$, which we may take as the origin in $\mathbb{R}^{m+1}$, we may choose a coordinate system $t_1, \cdots, t_m, x$ in $\mathbb{R}^{m+1}$ so that $L$ is spanned by $m$ linearly independent vector fields $L_1, \cdots, L_m$ satisfying

\begin{equation}
L_j = \frac{\partial}{\partial t_j} + \lambda_j(t, x) \frac{\partial}{\partial x}, \quad j = 1, \cdots, m,
\end{equation}

where $\lambda_j$ are (real-) analytic complex valued functions in $\Omega$.

Let $Z(t, x)$ be the unique analytic solution of the Cauchy problem

\begin{equation}
L_j Z = 0, \quad j = 1, \cdots, m \quad Z|_{t=0} = x \quad \text{in} \quad \Omega
\end{equation}

which may take the form, after shrinking $\Omega$,

\begin{equation}
Z = x + i\phi(t, x), \quad \phi \text{ real valued, } \phi(0, x) = 0.
\end{equation}

We say that $Lu = 0, u \in D'(\Omega)$ if $Mu = 0$ for all vector field $M$ in $L$, which is equivalent to saying that $L_j u = 0$ in $\Omega$ for all $j = 1, \cdots, m$.

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2. Pointwise-hypoellipticity

Definition 2.1. The Lie algebra $L$ is analytic hypoelliptic at a point $w_0$ of $\Omega$ if given any distribution $u$ in an open neighborhood $U \subset \Omega$ of $w_0$ such that, given any vector field $M \in L$, $Mu$ is analytic in some open neighborhood $V \subset U$ of $w_0$, then $u$ is analytic in some open neighborhood $U_0 \subset U$ of $w_0$.

We shall say that $L$ is analytic hypoelliptic in a subset $A$ of $\Omega$ if this is so at every point of $A$, which is the same as the usual analytic hypoellipticity, when $A$ is open. We shall also use the $C^\omega$ analogues of these definitions.

In [1], M.S. Baouendi and F. Treves characterized the pointwise-analytic hypoellipticity of $L$ by using a local constancy principle of $L$. Let $r$ be a positive number and $J$ be an open interval containing the origin such that the closure $\overline{B_r \times J}$ of $B_r \times J$, $B_r = \{ t \in \mathbb{R}^n : |t| < r \}$, is contained in $\Omega$.

Proposition 2.1. (Baouendi and Treves [1]) The following properties of $L$ are equivalent.

(A) $L$ is not analytic hypoelliptic at the origin.

(B) There is a number $r', 0 < r' < r$ such that zero is not an interior point of the image of $B_{r'}$ under the map $t \mapsto \phi(t, 0)$.

(C) There is an open neighborhood $V \subset \Omega$ of the origin such that, given any integer $k \geq 0$, there is an $u \in C^k(V)$ satisfying $Lu = 0$, but not of class $C^{a(1)}$ in any subneighborhood of the origin.

In particular, if $L$ is hypoelliptic at a point in $\Omega$, it is analytic hypoelliptic at the same point. But there is an example of a single vector field in $\mathbb{R}^2$ which is analytic hypoelliptic but not hypoelliptic at the origin [1].

Theorem 2.1. If $L$ is hypoelliptic at a point in $\Omega$, then it is analytic hypoelliptic in some neighborhood of the same point.

Proof. It suffices to show the theorem at the origin. If we assume that any neighborhood of the origin contains a point at which $L$ is not analytic hypoelliptic, then there is a sequence ${w^i}$, $w^i = (t^i, x^i)$, of points in $\Omega$ which converges to the origin
and at which $L$ is not analytic hypoelliptic. Since $L$ is not analytic hypoelliptic at each $w^i$, by proposition 2.1, there is a positive number $r^i$, such that $\phi(w^i)$ is not an interior point of the image of $B_{r^i}$, open ball with center $t^i$ and radius $r^i$, under the map $t \mapsto \phi(t, x^i)$. So if $J^i$ is an open interval containing $x^i$ such that $\overline{B_{r^i} \times J^i} \subset \Omega$, then $Z(B_{r^i} \times J^i)$ does not intersect at least one of the two open rays $Re Z = x^i$, $Im Z > \phi(w^i)$, and $Re Z = x^i$, $Im Z < \phi(w^i)$, so that one can define a single valued branch of $Z^{3/2}$ on $Z(B_{r^i} \times J^i)$. For each $i$, select a function $g_i \in C_0^\infty(B_{r^i} \times J^i)$ equal to one in a neighborhood of $w^i$ and whose support does not contain any point $w^j$ for $j \neq i$. The continuous function $u_i(t, x) = g_i(t, x)[Z(t, x) - Z(w^i)]^{3/2}$ satisfies $L \mu_i \in C_0^\infty(R^{n+1})$, for all $j = 1, \ldots, m$. We can find a sequence of numbers $c_i > 0$ such that $\sum c_i \mu_i$ converges uniformly to a compactly supported continuous function $u$ and that $\sum c_i L \mu_i$ converges in $C_0^\infty(R^{n+1})$, for all $j = 1, \ldots, m$. But the singular support of $u$ must contain the origin since $u$ is singular at each $w^i$, whence $L$ cannot be hypoelliptic at the origin. This completes the proof.

If $m$ is equal to one, that is, if $L$ is generated by a single complex vector field, then due to the result in [3] the hypoellipticity and analytic hypoellipticity for $L$ are equivalent. So we have the following corollary:

**Corollary 2.1.** For any complex analytic vector field $L$ in $R^n$, the followings are equivalent.

(A) $L$ is hypoelliptic at the origin.

(B) There is a neighborhood of the origin in which $L$ is analytic hypoelliptic.

(C) There is a neighborhood of the origin in which $L$ is hypoelliptic.

**References**


KAIST
P.O. Box 150, Cheongryang
Seoul 130—650, Korea