Bull. Korean Math. Soc. 26(1989), No. 1, pp. 53~67

CONVERGENCE AND GENERAL REPRESENTATION OF THE EXPONENTIALLY BOUNDED C-SEMIGROUPS IN BANACH SPACE

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1. Introduction

In this paper we consider the exponentially bounded C-semigroups in Banach space. Let X be a Banach space and $C: X \to X$ be an injective bounded linear operator with dense range. A family $\{S(t)\,;\,t \geqq 0\}$ of bounded linear operators from X into itself is called an exponentially bounded C-semigroup if

(1) $S(t+s)C = S(t)S(s)$ for t, $s \geqq 0$ and $S(0) = C$,

(2) there exists $M \geqq 0$ and $a \geqq 0$ such that $\|S(t)\| \leqq Me^{at}$ for $t \geqq 0$,

(3) for every $x \in X$, $S(t)x$ is continuous in $t \geqq 0$.

For every $t \geqq 0$, let $T(t)$ be the closed linear operator defined by $T(t)x = C^{-1}S(t)x$ for $x \in D(T(t)) = \{x \in X\,;\, S(t)x \in R(C)\}$. We define the operator G by

$$D(G) = \{x \in R(C)\,;\, \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists}\}$$

and

$$Gx = \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ for each } x \in D(G). \tag{1.1}$$

It is known that [2, 5] that G is densely defined and closable. The closure $\bar{G}$ is called C-c. i. g. (C-complete infinitesimal generator) of $\{S(t)\,;\,t \geqq 0\}$.

For every $\lambda > a$, define the bounded linear operator $L_\lambda : X \to X$ by

$$L_\lambda x = \int_0^\infty e^{-\lambda t} S(t)x\,dt \text{ for } x \in X. \tag{1.2}$$

It is known [2, 5] that L_λ is injective for all $\lambda > a$ and the closed linear operator Z defined by

$$Zx = L_\lambda^{-1}(\lambda L_\lambda - C)x = (\lambda - L_\lambda^{-1}C)x \tag{1.3}$$

with $D(Z) = \{x \in X\,;\, Cx \in R(L_\lambda)\}$ is independent of $\lambda > a$.

Received July 16, 1988.

The operator Z is called the generator of $\{S(t)\ ;\ t\geqq 0\}$. We denote $D(A)$, $R(A)$ are domain and range of A respectively and I is identity operator.

In §2, we deal with the convergence of exponentially bounded C-semigroups and §3 treats general representation of the exponentially bounded C-semigroups in Banach space.

2. The convergence of the exponentially bounded C-semigroups

We start the following generation theorem for the exponentially bounded C-semigroups.

THEOREM 2.1. ([6, Theorem 2.1]) *An operator A is the C-c.i.g. of an exponentially bounded C-semigroup $\{S(t)\ ;\ t\geqq 0\}$ with $\|S(t)\|\leqq Me^{at}$ if and only if*

(A) *A is densely defined closed linear operator in X,*

(B) *λ-A is injective for $\lambda>a$,*

(C) *$D((\lambda-A)^{-n})\supset R(C)$ for $n\geqq 1$ and $\lambda>a$,*

(D) *$(\lambda-A)^{-1}Cx=C(\lambda-A)^{-1}x$ for $x\in D((\lambda-A)^{-1})$ and $\lambda>a$,*

(E) *$\|(\lambda-A)^{-n}C\|\leqq M/(\lambda-a)^n$ for $n\geqq 1$ and $\lambda>a$,*

(F) *$CD(A)$ is a core of A.*

THEOREM 2.2. ([2, Theorem 11]) *Let A be a closed linear operator satisfying (A)$\sim$(E). Then there exists an exponentially bounded C-semigroup $\{S(t)\ ;\ t\geqq 0\}$ such that*

(A) *$\|S(t)\|\leqq Me^{at}$ for all $t\geqq 0$,*

(B) *$(\lambda-A)^{-1}Cx=\int_0^\infty e^{-\lambda t}S(t)xdt$ for $\lambda>a$ and $x\in X$.*

For each $\rho>0$, $\{S_\rho(t)\ ;\ t\geqq 0\}$ satisfy the conditions (1), (2), (3), then $\{S_\rho(t)\ ;\ t\geqq 0\}$ is called an exponentially bounded C_ρ-semigroup on X.

THEOREM 2.3. *Let for each $\rho>0$, $\{S_\rho(t)\ ;\ t\geqq 0\}$ be the exponentially bounded C_ρ-semigroup on X with the C_ρ-c.i.g. A_ρ and that $\|S_\rho(t)\|\leqq Me^{at}$, where M, a are independent of ρ. Suppose $\{S(t)\ ;\ t\geqq 0\}$ is an exponentially bounded C-semigroup on X, A is the C-c.i.g. of $\{S(t)\ ;\ t\geqq 0\}$ and $\lim_{\rho\to 0} C_\rho x=Cx$ for $x\in X$. Then* *the following assertions are equivalent.*

(a) *For every* $x\in X$ *and* λ *with* $Re\,\lambda>a$,

$(\lambda-A_\rho)^{-1}C_\rho x\to(\lambda-A)^{-1}Cx$ *as* $\rho\to 0$.

(b) *For every* $x\in X$ *and* $t\geqq 0$, $S_\rho(t)x\to S(t)x$ *as* $\rho\to 0$.

Moreover, the convergence in part (b) is uniform on bounded t-intervals.

Proof. (a)$\Longrightarrow$(b)

For every $x\in X$ and λ with $\mathrm{Re}\,\lambda>a$, the X-valued function $s\to S_\rho(t-s)(\lambda-A_\rho)^{-1}C_\rho S(s)(\lambda-A)^{-1}Cx$ is differentiable. Since $\frac{d}{dt}S(t)x=S(t)\bar{G}x$ [2, Lemma 8] for $x\in D(\bar{G})$ and $\bar{G}(\lambda-\bar{G})^{-1}C(=\lambda(\lambda-\bar{G})^{-1}C-C)$ commute with $S(s)$ [4, Theorem 2.2], where $\bar{G}$ is C-c. i. g. which corresponding the exponentially bounded C-semigroup of $\{S(t)\ ;\ t\geqq 0\}$, we have

$$\frac{d}{ds}[S_\rho(t-s)(\lambda-A_\rho)^{-1}C_\rho S(s)(\lambda-A)^{-1}Cx]$$

$$\begin{aligned}&=S_\rho(t-s)(\lambda-A_\rho)^{-1}C_\rho S(s)A(\lambda-A)^{-1}Cx\\&\quad-S_\rho(t-s)A_\rho(\lambda-A_\rho)^{-1}C_\rho S(s)(\lambda-A)^{-1}Cx\\&=S_\rho(t-s)(\lambda-A_\rho)^{-1}C_\rho S(s)(\lambda(\lambda-A)^{-1}C-C)x\\&\quad-S_\rho(t-s)(\lambda(\lambda-A_\rho)^{-1}C_\rho-C_\rho)S(s)(\lambda-A)^{-1}Cx\\&=S_\rho(t-s)[C_\rho(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho C]S(s)x.\end{aligned}$$

Using $S_\rho(0)=C_\rho$, $S(0)=C$ and integrating the above equality,

$$(2.1)\quad (\lambda-A_\rho)^{-1}C_\rho[C_\rho S(t)-S_\rho(t)C](\lambda-A)^{-1}Cx$$
$$=\int_0^t S_\rho(t-s)[C_\rho(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho C]\ S(s)xds$$

for all $x\in X$, $\mathrm{Re}\,\lambda>a$ and $0\leqq s\leqq t$. Fix $x\in X$, we see that

$$\begin{aligned}(2.2)\quad &\|(S_\rho(t)C-C_\rho S(t))(\lambda-A)^{-1}Cx\|\\&\leqq\|S_\rho(t)((\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho)Cx\|+\|(\lambda-A_\rho)^{-1}\\&\quad C_\rho(S_\rho(t)C-C_\rho S(t))x\|+\|C_\rho((\lambda-A_\rho)^{-1}C_\rho-(\lambda-A)^{-1}C)S(t)x\|\end{aligned}$$

for $0\leqq t\leqq T$.

By condition (a) we obtain

$$(2.3)\quad \|S_\rho(t)[(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho]Cx\|\to 0 \text{ as } \rho\to 0$$

and

$$(2.4)\quad \|C_\rho[(\lambda-A_\rho)^{-1}C_\rho-(\lambda-A)^{-1}C]S(t)x\|\to 0 \text{ as } \rho\to 0$$

uniformly on $[0, T]$. From (2.1) we have

$$\|(\lambda-A_\rho)^{-1}C_\rho[C_\rho S(t)-S_\rho(t)C](\lambda-A)^{-1}Cx\|$$
$$\leq\int_0^t\|S_\rho(t-s)\|\ \|[C_\rho(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho C]S(s)x\|ds$$
$$\leq\int_0^T\|S_\rho(t-s)\|\ \|[C_\rho(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho C]S(s)x\|ds$$

Since $\|S_\rho(t-s)\|\ \|[C_\rho(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho C]S(s)x\|$
$\leq 2M^4e^{at}(\lambda-a)^{-1}\|x\|$

and

$$\|S_\rho(t-s)\|\ \|[C_\rho(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho C]\ S(s)x\|\to 0 \text{ as } \rho\to 0,$$

we obtain

$$\int_0^T\|S_\rho(t-s)\|\ \|[C_\rho(\lambda-A)^{-1}C-(\lambda-A_\rho)^{-1}C_\rho C]S(s)x\|\to 0 \text{ as } \rho\to 0$$

by Lebesgues's bounded convergence theorem. Therefore

$$(2.5)\qquad \lim_{\rho\to 0}\|(\lambda-A_\rho)^{-1}C_\rho[C_\rho S(t)-S_\rho(t)C](\lambda-A)^{-1}Cx\|=0$$

and the limit in (2.5) is uniform on $[0, T]$. Since every $z\in CD(A)$ can be written as $z=(\lambda-A)^{-1}Cx=C(\lambda-A)^{-1}x$ for some $x\in D(\lambda-A)^{-1})$. From (2.5), for $z\in CD(A)$,

$$(2.6)\qquad \lim_{\rho\to 0}\|(\lambda-A_\rho)^{-1}C_\rho[C_\rho S(t)-S_\rho(t)C]x\|=0.$$

By (2.2), (2.3), (2.4) and (2.6)

$$(2.7)\quad \lim_{\rho\to 0}\|[S_\rho(t)C-C_\rho S(t)](\lambda-A)^{-1}Cx\|=0 \text{ for } x\in CD(A)$$

and the limit (2.7) is uniform on $[0, T]$.

Since $CD(A)$ is dense in X, it follows that (2.7) hold for every $x\in X$ uniformly on $[0, T]$. In other words, for $z\in CD(A)$,

$$\lim_{\rho\to 0}\|S_\rho(t)C-C_\rho(S(t))z\|=0.$$

Hence $S_\rho(t)x\to S(t)x$ as $\rho\to 0$ for all $x\in X$ and $t\geq 0$.

(b)⇒(a)

Now, assume that (b) hold and $\mathrm{Re}\,\lambda>a$ then

$$(2.8)\quad \|(\lambda-A_\rho)^{-1}C_\rho x-(\lambda-A)^{-1}Cx\|\leq\int_0^\infty e^{-(Re\lambda)t}\|(S_\rho(t)x-S(t))x\|dt.$$

The right-hand side of (2.8) tends to zero as $\rho\to 0$ by (b) and Lebesgue's dominated convergence theorem and therefore (b)⇒(a).

THEOREM 2.4. *Let* $\{S_\rho(t)\ ;\ t\geq 0\}$ $(\rho>0)$ *be an exponentially bounded* C_ρ*-semigroup with* $\|S_\rho(t)\|\leq Me^{at}$ *for* $t\geq 0$ *and* A_ρ *be the* C_ρ*-c.i.g. of* $\{S_\rho(t)\ ;\ t\geq 0\}$. *Let* λ_0 *satisfy* $Re\,\lambda_0>a$, *suppose that*

(a) *there exists an injective bounded linear operator C with dense range such that* $\lim_{\rho\to 0} C_\rho x = Cx$ *for* $x \in X$,

(b) $\lim_{\rho\to 0} (\lambda_0 - A_\rho)^{-1} C_\rho x = L(\lambda_0)x$ *for* $x \in X$,

(c) *the set* $\{x \in X\,;\, Cx \in R(L(\lambda_0))\}$ *is dense in* X.

Then there exists a C-c.i.g A and its corresponding C-semigroup $\{S(t)\,;\, t \geqq 0\}$ *with* $\|S(t)\| \leqq Me^{at}$ *for* $t \geqq 0$ *such that* $L(\lambda_0) = (\lambda_0 - A)^{-1}C$ *and* $\lim_{\rho\to 0} S_\rho(t)x = S(t)x$ *for* $x \in X$ *uniformly for t in bounded intervals in* $[0, \infty)$.

Proof. From (b), the estimate $\|(\lambda_0 - A_\rho)^{-n} C_\rho\| \leqq M(\mathrm{Re}\,\lambda_0 - a)^{-1}$ and the expansion

$$(\lambda - A_\rho)^{-1} C_\rho = \sum_{k=0}^{\infty} (\lambda_0 - \lambda)^k (\lambda_0 - A_\rho)^{-(k+1)} C_\rho.$$

We deduce that

$$\lim_{\rho\to 0} (\lambda - A_\rho)^{-1} C_\rho x = L(\lambda)x \tag{2.9}$$

exists for all $x \in X$ and $a < \mathrm{Re}\,\lambda < 2\,\mathrm{Re}\,\lambda_0 - a$. Also, From (2.9),

$$\|(\lambda - A_\rho)^{-n} C_\rho\| \leqq M/(\mathrm{Re}\,\lambda - a)^n$$

and

$$(\lambda' - A_\rho)^{-1} C_\rho = \sum_{k=0}^{\infty} (\lambda - \lambda')^k (\lambda - A_\rho)^{-(k+1)} C_\rho,$$

there exists

$$\lim_{\rho\to 0} (\lambda' - A_\rho)^{-1} C_\rho x = L(\lambda')x$$

for all $x \in X$ and $a < \mathrm{Re}\,\lambda' < 4\,\mathrm{Re}\,\lambda_0 - 3a$. Proceeding by induction we obtain (2.9) holds for all $x \in X$ and $a < \mathrm{Re}\,\lambda < \infty$. Clearly $L(\lambda)C = CL(\lambda)$ and

$$(\lambda - \mu) L(\lambda) L(\mu) = L(\mu)C - L(\lambda)C \quad \text{for } \mathrm{Re}\,\lambda,\ \mathrm{Re}\,\mu > a. \tag{2.10}$$

Next, we will prove that $N(L(\lambda)) = \{0\}$ for $\mathrm{Re}\,\lambda > a$, where $N(L(\lambda))$ is the null set of $L(\lambda)$. From the definition of $L(\lambda)$, we have

$$\|(\lambda - a) L(\lambda)x\| \leqq \underline{\lim}_{\rho\to 0} \|(\lambda - a)(\lambda - A_\rho)^{-1} C_\rho x\| \leqq M\|x\| \tag{2.11}$$

for all $x \in X$ and $\mathrm{Re}\,\lambda > a$, which implies $\lim_{\lambda\to\infty} \|L(\lambda)\| = 0$. By (2.10) and this

$$\begin{aligned}
&\|[(\lambda - a)L(\lambda) - C]L(\lambda_0)\| \\
&= \|(\lambda - \lambda_0)L(\lambda)L(\lambda_0) + (\lambda_0 - a)L(\lambda)L(\lambda_0) - CL(\lambda_0)\| \\
&= \|L(\lambda)[(\lambda_0 - a)L(\lambda_0) - C]\|
\end{aligned}$$

$$=\|L(\lambda)\|\ \|(\lambda_0-a)L(\lambda_0)-C\|$$
$$\to 0 \text{ as } \lambda\to\infty$$

i.e., $\lim_{\lambda\to\infty}(\lambda-a)L(\lambda)x=C(x)$ for $x\in R(L(\lambda_0))$. Let $x\in\{x\in X;\ Cx\in R(L(\lambda_0))\}$ and $Cx=L(\lambda_0)y$ for some $y\in X$. Then

$$\lim_{\lambda\to\infty}(\lambda-a)L(\lambda)Cx=\lim_{\lambda\to\infty}(\lambda-a)L(\lambda)L(\lambda_0)y=CL(\lambda_0)y=C^2x$$

i.e., $\lim_{\lambda\to\infty}(\lambda-a)L(\lambda)x=Cx$ for $x\in C\{x\in X;\ Cx\in R(L(\lambda_0))\}$.

From (c) and (2.11), we have

(2.12) $$\lim_{\lambda\to\infty}(\lambda-a)L(\lambda)x=Cx \text{ for } x\in X.$$

Let $x\in N(L(\lambda))$. Since $(\lambda-\mu)L(\mu)L(\lambda)x=L(\mu)Cx-L(\lambda)Cx$ for $\operatorname{Re}\mu$, $\operatorname{Re}\lambda>a$, we have $N(L(\lambda))=N(L(\mu))$, which yields $(\mu-a)L(\mu)x=0$ for every $\operatorname{Re}\mu>a$. Therefore (2.12) implies $Cx=0$ and hence $x=0$ by the injectivity of C. Thus $N(L(\lambda))=\{0\}$.

Define a closed linear operator A by

$$Ax=(\lambda_0-L(\lambda_0)^{-1}C)x \text{ for } x\in D(A),$$

where $D(A)=\{x\in X;\ Cx\in R(L(\lambda_0))\}$ $(=\{x\in X;\ Cx\in R(L(\lambda))\})$ is independent of λ. By assumption (c), A is densely defined operator. From the definition of A, $(\lambda_0-A)L(\lambda_0)x=Cx$ for $x\in X$ and $L(\lambda_0)(\lambda_0-A)x=Cx$ for $x\in D(A)$.

From this and (2.10)

$$\begin{aligned}(\lambda-A)L(\lambda)Cx&=[(\lambda-\lambda_0)+(\lambda_0-A)]L(\lambda)Cx\\&=[(\lambda-\lambda_0)+(\lambda_0-A)]L(\lambda_0)[C-(\lambda-\lambda_0)L(\lambda)]x\\&=(\lambda_0-A)L(\lambda_0)Cx+(\lambda-\lambda_0)[L(\lambda_0)C\\&\quad-(\lambda-\lambda_0)L(\lambda)L(\lambda_0)-(\lambda_0-A)L(\lambda_0)L(\lambda)]x\\&=C^2x+(\lambda-\lambda_0)[L(\lambda_0)C-(\lambda-\lambda_0)L(\lambda)L(\lambda_0)\\&\quad-L(\lambda)C]x\\&=C^2x\end{aligned}$$

for $x\in X$ and $\operatorname{Re}\lambda>a$. Since $(\lambda-A)L(\lambda)$, C are bounded in X and $R(C)$ is dense in X,

(2.13) $$(\lambda-A)L(\lambda)x=Cx \text{ for } x\in X.$$

On the other hand

$$\begin{aligned}CL(\lambda)(\lambda-A)x&=L(\lambda)C(\lambda-A)x\\&=L(\lambda_0)[C-(\lambda-\lambda_0)L(\lambda)][(\lambda-\lambda_0)+(\lambda_0-A)]x\\&=C^2x+(\lambda-\lambda_0)[L(\lambda_0)C-(\lambda-\lambda_0)L(\lambda)L(\lambda_0)-L(\lambda)C]x\\&=C^2x\end{aligned}$$

for $x\in D(A)$. From the injectivity of C, we get

(2.14) $$L(\lambda)(\lambda-A)x=Cx \text{ for } x\in D(A).$$

From (2.13), (2.14), $\lambda-A$ is injective and $(\lambda-A)^{-1}Cx=C(\lambda-A)^{-1}x$ for $x\in D((\lambda-A)^{-1})$ and $\mathrm{Re}\,\lambda>a$. Let $\mathrm{Re}\,\lambda>a$. Again using (2.13) and (2.14),

(2.15) $$D((\lambda-A)^{-m})\supset R(C^m) \text{ and } (\lambda-A)^{-m}C^m z=L(\lambda)^m z \text{ for } m\geqq 1 \text{ and } z\in X.$$

Since $\|(\lambda-A)^{-m}C^m z\|=\|L(\lambda)^m z\|=\lim_{\rho\to 0}\|(\lambda-A_\rho)^{-m}C_\rho^m z\|\leqq(M/(\mathrm{Re}\lambda-a)^m)\|C^{m-1}z\|$ for $m\geqq 1$ and $z\in X$,

(2.16) $$\|(\lambda-A)^{-m}Cx\|\leqq(M/(\mathrm{Re}\lambda-a)^m)\|x\| \text{ for } x\in R(C^{m-1}).$$

Using (2.15), (2.16) and the closeness of $(\lambda-A)^{-1}$, we have

(2.17) $D((\lambda-A)^{-m}\supset R(C)$, $\|(\lambda-A)^{-m}C\|\leqq M/(\mathrm{Re}\,\lambda-a)^m$ and $(\lambda-A)^{-m}C$ is bounded linear operator for $m\geqq 1$ and $\mathrm{Re}\,\lambda>a$.

In fact, clearly (2.17) holds for $m=1$. Suppose (2.17) is true for $m=k$. Let $x\in X$. Since $R(C^k)$ is dense in X, there is $x_\rho\in R(C^k)$ such that $x_\rho\to x$ as $\rho\to 0$, which implies $\lim_{\rho\to 0}(\lambda-A)^{-k}Cx_\rho=(\lambda-A)^{-k}Cx$. By (2.16)

$$\|(\lambda-A)^{-(k+1)}Cx_\rho-(\lambda-A)^{-(k+1)}Cx_\mu\|\leqq(M/(\mathrm{Re}\,\lambda-a)^{k+1})\|x_\rho-x_\mu\|\to 0$$

as $\rho,\mu\to 0$. Hence $(\lambda-A)^{-1}[(\lambda-A)^{-k}Cx_\rho]=(\lambda-A)^{-(k+1)}Cx_\rho$ is convergent as $\rho\to 0$. i.e., $Cx\in D((\lambda-A)^{-(k+1)})$. Therefore $D((\lambda-A)^{-(k+1)})\supset R(C)$ and $(\lambda-A)^{-(k+1)}C$ is closed linear operator on X then it is bounded on X by closed graph theorem. So $D((\lambda-A)^{-m})\supset R(C)$ and $(\lambda-A)^{-m}C$ is bounded linear operator. Therefore, by Theorem 2.2 in §2, there exists an exponentially bounded C-semigroup $\{S(t)\ ;\ t\geqq 0\}$ satisfying

$$(\lambda-A)^{-1}Cx=\int_0^\infty e^{-\lambda t}S(t)x\,dt \text{ for } \lambda>a \text{ and } x\in X.$$

Since $L(\lambda)=(\lambda-A)^{-1}Cx$ for $x\in X$, A is the generator of a C-semigroup $\{S(t)\ ;\ t\geqq 0\}$ by the definition of A. Therefore, if A is the C-c.i.g. of $\{S(t)\ ;\ t\geqq 0\}$ then our assertion holds by Theorem 2.3.

THEOREM 2.5. *Let* $\{S_\rho(t)\ ;\ t\geqq 0\}\ (\rho>0)$ *be an exponentially bounded C-semigroup with* $\|S_\rho(t)\|\leqq Me^{at}$ *for* $t\geqq 0$ *and* A_ρ *be the* C_ρ-*c.i.g. of* $\{S_\rho(t)\ ;\ t\geqq 0\}$ *respectively. Let* λ_0 *statisy* $\mathrm{Re}\,\lambda_0>a$ *and*

(a) *there exists an injective bounded linear operator C with dense range such that* $\lim_{\rho\to 0} C_\rho x = Cx$ *for* $x \in X$,

(b) $\lim_{\rho\to 0} A_\rho x = Ax$ *for* $x \in D$, *where D is a dense subset of X*,

(c) $D((\lambda_0 - A)^{-1}) \supset R(C)$.

Then A is closable and $\bar{A}$ is the C-c.i.g. of an exponentially bounded C-semigroup. Moreover, if $\{S(t)\,;\, t \geqq 0\}$ is the exponentially bounded C-semigroup generated by $\bar{A}$ then for all $t \geqq 0$ and $x \in X$,

$$\lim_{\rho\to 0} S_\rho(t)x = S(t)x \tag{2.18}$$

with $\|S(t)\| \leqq Me^{ta}$ *and the limit* (2.18) *is uniform for t in bounded intervals on* $[0, \infty)$.

Proof. Let $y \in D$, $x = (\lambda_0 - A)y$ and $x_\rho = (\lambda_0 - A_\rho)y$. Since $A_\rho y \to Ay$, $x_\rho \to x$. Also, since $\|(\lambda_0 - A_\rho)^{-1} C_\rho\| \leqq M/(\lambda_0 - a)$, it follows that

$$\begin{aligned} \lim_{\rho\to 0}(\lambda_0 - A_\rho)^{-1} C_\rho x &= \lim_{\rho\to 0}((\lambda_0 - A_\rho)^{-1} C_\rho (x - x_\rho) + C_\rho y) \\ &= Cy \end{aligned} \tag{2.19}$$

i.e., $(\lambda_0 - A_\rho)^{-1} C_\rho$ converges on the range of $\lambda_0 - A$. By (c) and the density of $R(C)$, $R(\lambda_0 - A)$ is dense in X and $\|(\lambda_0 - A_\rho)^{-1} C_\rho\|$ are uniformly bounded by assumption. Therefore $(\lambda_0 - A_\rho)^{-1} C_\rho x$ converges for every $x \in X$. Let

$$\lim_{\rho\to 0}(\lambda_0 - A_\rho)^{-1} C_\rho x = L(\lambda_0)x. \tag{2.20}$$

From (2.19), we have $Cy = L(\lambda_0)x$ for all $y \in D$. Thus $D \subset \{y \in X\,;\, Cy \in R(L(\lambda_0))\}$. Since D is dense subset of X, $\{y \in X\,;\, Cy \in R(L(\lambda_0))\}$ is dense in X. By Theorem 2.4 there exists C-c.i.g. A' of an exponentially bounded C-semigroup such that $L(\lambda_0) = (\lambda_0 - A')^{-1} C$. To conclude the proof we must show that $\bar{A} = A'$. For any $x \in D$,

$$\lim_{\rho\to 0}(\lambda_0 - A_\rho)^{-1} C_\rho (\lambda_0 - A)x = (\lambda_0 - A')^{-1} C(\lambda_0 - A)x. \tag{2.21}$$

On the other hand, since $\|(\lambda_0 - A_\rho)^{-1} C_\rho\|$ are uniformly bounded and $\lim_{\rho\to 0} A_\rho x = Ax$ for $x \in D$ and

$$\begin{aligned} &\lim_{\rho\to 0}(\lambda_0 - A_\rho)^{-1} C_\rho (\lambda_0 - A)x \\ &= \lim_{\rho\to 0}(\lambda_0 - A_\rho)^{-1} C_\rho (\lambda_0 - A_\rho)x + \lim_{\rho\to 0}(\lambda_0 - A_\rho)^{-1} C_\rho (A_\rho - A)x \\ &= Cx. \end{aligned}$$

Therefore $(\lambda_0-A')^{-1}C(\lambda_0-A)x=Cx$ for $x\in D$. From $(\lambda_0-A')^{-1}C=C(\lambda_0-A')^{-1}$ and injectivity of C,

(2.22) $$(\lambda_0-A')^{-1}(\lambda_0-A)x=x$$

and this implies $A'x=Ax$ for $x\in D$. Thus $A'\supset A$. Since A' is closed, A is closable. Next we show that $\bar{A}\supset A'$. Let $y'=A'x'$. Since $R(\lambda_0-A)$ is dense in X and $R(\lambda_0-A)\supset R(C)$, there exists a sequence $x_\rho\in D$ such that

(2.23) $Cy_\rho=(\lambda_0-A')x_\rho=(\lambda_0-A)x_\rho\to\lambda_0x'-y'=(\lambda_0-A')x'$ as $\rho\to 0$.

By (2.23),

(2.24) $x_\rho=(\lambda_0-A')^{-1}Cy_\rho\to(\lambda_0-A')^{-1}(\lambda_0-A')x'$ as $\rho\to 0$

and it follows that

(2.25) $Ax_\rho=\lambda_0x_\rho-Cy_\rho\to\lambda_0x'-(\lambda_0x'-A'x')=y'$ as $\rho\to 0$.

From (2.24) and (2.25), $y'=\bar{A}x'$ and $\bar{A}\supset A'$. Thus $\bar{A}=A'$. The rest of the assertions of the theorem now directly from Theorem 2.4.

3. General representation of exponentially bounded C-semigroup

Throughout this section $\{F(\rho)\ ;\ \rho\geqq 0\}$ is denote a family of closed linear operator and $C_\rho\ ;\ X\to X$ are injective bounded linear operators with dense range for each ρ. We consider the following conditions:

(i) $\lim_{\rho\to 0}\rho^{-1}(F(\rho)x-x)=Ax$ for $x\in D$, where D is dense subset of X,

(ii) there are $M>0$ and $a\geqq 0$ such that $\|F(\rho)^kC_\rho\|\leqq Me^{a\rho k}$ for $k=1,2,\cdots$,

(iii) $F(\rho)C_\rho x=C_\rho F(\rho)x$ for $x\in R(C_\rho)$,

(iv) $D(F(\rho)^k)\supset R(C_\rho)$.

Lemma 3.1. *Let $\{F(\rho)\ ;\ \rho\geqq 0\}$ satisfying (ii)$\sim$(iv). Then for $x\in X$ and $m=1,2,\cdots$, we have*

(3.1) $$\|e^{-m}\sum_{k=0}^{\infty}\frac{m^k}{k!}F(\rho)^kC_\rho x-F(\rho)^mC_\rho x\|$$
$$\leqq M\,exp(a\rho(m-1)+(e^{a\rho}-1)m)\{m^2(e^{a\rho}-1)^2+me^{a\rho}\}^{\frac{1}{2}}\|F(\rho)x-x\|.$$

Proof. We note that by (ii) and boundedness of C_ρ,

$\sum_{k=0}^{\infty}\frac{m^k}{k!}F(\rho)^k C_\rho$ converges in norm and define a bounded linear operator. Let $x\in X$. If $k\geqq m$ then

$$\begin{aligned}\|F(\rho)^k C_\rho x - F(\rho)^m C_\rho x\| &= \|\sum_{j=m}^{k-1} F(\rho)^j C_\rho (F(\rho)x - x)\| \\ &\leqq \sum_{j=m}^{k-1}\|F(\rho)^j C_\rho\|\|F(\rho)x - x\| \\ &\leqq M\sum_{j=m}^{k-1} e^{a\rho j}\|F(\rho)x - x\| \\ &\leqq M e^{a\rho(k-1)}(k-m)\|F(\rho)x - x\|\end{aligned}$$

by (ii) and (iii).

On the other hand, for $k<m$,

(3.3) $\|F(\rho)^k C_\rho x - F(\rho)^m C_\rho x\| \leqq M e^{a\rho(m-1)}(m-k)\|F(\rho)x - x\|$.

From (3.2) and (3.3), we have

$$\|F(\rho)^k C_\rho x - F(\rho)^m C_\rho x\| \leqq M e^{a\rho(m-1)+a\rho k}|k-m|\,\|F(\rho)x - x\|.$$

Therefore, we obtain that

$$\begin{aligned}(3.4)\quad &\|e^{-m}\sum_{k=0}^{\infty}\frac{m^k}{k!}F(\rho)^k C_\rho x - F(\rho)^m C_\rho x\| \\ &= \|e^{-m}\sum_{k=0}^{\infty}\frac{m^k}{k!}(F(\rho)^k C_\rho x - F(\rho)^m C_\rho x)\| \\ &\leqq e^{-m}\sum_{k=0}^{\infty}\frac{m^k}{k!}\|F(\rho)^k C_\rho x - F(\rho)^m C_\rho x\| \\ &\leqq M e^{a\rho(m-1)} e^{-m}\sum_{k=0}^{\infty}\frac{|k-m|e^{a\rho k}m^k}{k!}\|F(\rho)x - x\|.\end{aligned}$$

Using the Schwarz iniquality we have

$$\begin{aligned}(3.5)\quad \sum_{k=0}^{\infty}\frac{|k-m|e^{a\rho k}m^k}{k!} &= \sum_{k=0}^{\infty}\left(\frac{e^{a\rho k}m^k}{k!}\right)^{\frac{1}{2}}\left(\frac{(k-m)^2 e^{a\rho k}m^k}{k!}\right)^{\frac{1}{2}} \\ &\leqq \left(\sum_{k=0}^{\infty}\frac{e^{a\rho k}m^k}{k!}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\frac{(k-m)^2 e^{a\rho k}m^k}{k!}\right)^{\frac{1}{2}} \\ &= \exp(e^{a\rho}m)\{m^2(e^{a\rho}-1)^2 + me^{a\rho}\}^{\frac{1}{2}}.\end{aligned}$$

Combining (3.4) and (3.5) we obtain (3.1).

THEOREM 3.2. *Let $\{F(\rho)\,;\,\rho\geqq 0\}$ satisfy (i)∼(iv) and let $Re\,\lambda_0 > a$. We suppose that*

(a) there is an injective bounded linear operator C with dense range such that $\lim_{\rho\to 0} C_\rho x = Cx$ for $x\in X$,

(b) $R((\lambda_0-A)^{-1})\supset R(C)$.

Then there exists an exponentially bounded C_ρ*-semigroup* $\{S_\rho(t)\ ;\ t\geqq 0\}$ *with* C_ρ*-c.i.g.* $A_\rho=\rho^{-1}(F(\rho)-I)$, A *is closable and that* $\overline{A}$ *is the C-c.i.g. of an exponentially bounded C-semigroup. Moreover, if* $\{S(t)\ ;\ t\geqq 0\}$ *is the exponentially bounded C-semigroup generated by* $\overline{A}$, *then for every sequence of positive integers* $k_n\to\infty$ *satisfying* $k_n\rho_n\to t$ *we have*

(3.6) $\lim_{n\to\infty} F(\rho_n)^{k_n}C_{\rho_n}x=\lim_{n\to\infty} S_{\rho_n}(\rho_n k_n)x=S(t)x$ *for* $x\in X$

and $\|S(t)\|\leqq Me^{at}$ *for* $t\geqq 0$. *Choosing* $\rho_n k_n=t$ *for every* n, *the limit* (3.6) *is uniform in bounded* t*-intervals.*

Proof. We define $S_\rho(t)x=e^{-\frac{t}{\rho}}\sum_{k=0}^{\infty}\left(\frac{t}{\rho}\right)^k\frac{F(\rho)^k}{k!}C_\rho x$ for $x\in X$ and each $\rho>0$. Then $S_\rho(t)$ converges in norm by (ii) and closed linear operator for each $\rho>0$.

We first prove that $\{S_\rho(t)\ ;\ t\geqq 0\}$ satisfies C_ρ-semigroup property. Clearly $S_\rho(0)=C_\rho$. Let $\varepsilon>0$ be such that $\mathrm{Re}\,\lambda_0>a+\varepsilon$ and let $\rho_0>0$ be such that for $0<\rho\leqq\rho_0$, $(e^{a\rho}-1)\rho^{-1}<a+\varepsilon$.
Then we have

$$\begin{aligned}\|S_\rho(t)\|&\leqq e^{-\frac{t}{\rho}}\sum_{k=0}^{\infty}\left(\frac{t}{\rho}\right)^k\frac{\|F(\rho)^kC_\rho\|}{k!}\\&\leqq Me^{-\frac{t}{\rho}}\sum_{k=0}^{\infty}\left(\frac{t}{\rho}\right)^k\frac{e^{a\rho k}}{k!}\\&=M\exp\left(\frac{t}{\rho}(e^{\rho a}-1)\right)\\&\leqq Me^{(a+\varepsilon)t}\end{aligned}$$

for $0<\rho\leqq\rho_0$. For all $x\in X$ and t, $s\geqq 0$, we obtain that

$$\begin{aligned}S_\rho(t+s)C_\rho x&=e^{-\frac{1}{\rho}(t+s)}\sum_{k=0}^{\infty}\left(\frac{(t+s)}{\rho}\right)^k\frac{F(\rho)^k}{k!}C_\rho^2x\\&=e^{-\frac{1}{\rho}(t+s)}\sum_{k=0}^{\infty}\frac{1}{k!}\left\{\frac{t}{\rho}F(\rho)+\frac{s}{\rho}F(\rho)\right\}^kC_\rho^2x\\&=e^{-\frac{1}{\rho}(t+s)}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=0}^{k}\binom{k}{n}\left(\frac{t}{\rho}\right)^nF(\rho)^n\left(\frac{s}{\rho}\right)^{k-n}F(\rho)^{k-n}C_\rho^2x\\&=e^{-\frac{t}{\rho}}e^{-\frac{s}{\rho}}\sum_{m=0}^{\infty}\left(\frac{t}{\rho}\right)^m\frac{F(\rho)^m}{m!}C_\rho\sum_{k=0}^{\infty}\left(\frac{s}{\rho}\right)^k\frac{F(\rho)^k}{k!}C_\rho x\\&=S_\rho(t)S_\rho(s)x.\end{aligned}$$

Finally $\{S_\rho(t)\ ;\ t\geqq 0\}$ is strongly continuous since

$$
\begin{aligned}
&\|S_\rho(t)x-S_\rho(t+h)x\|\\
&=\|e^{-\frac{t}{\rho}}\sum_{k=0}^{\infty}\left(\frac{t}{\rho}\right)^k\frac{F(\rho)^k}{k!}C_\rho x-e^{-\frac{t}{\rho}}e^{-\frac{t}{\rho}}\sum_{k=0}^{\infty}\left(\frac{t+h}{\rho}\right)^k\frac{F(\rho)^k}{k!}C_\rho x\|\\
&\leqq e^{-\frac{t}{\rho}}\sum_{k=0}^{\infty}\frac{1}{k!}\left|\left(\frac{t}{\rho}\right)^k-e^{-\frac{h}{\rho}}\left(\frac{t+h}{\rho}\right)^k\right|\|F(\rho)^kC_\rho\|\|x\|\\
&\to 0 \text{ as } h\to 0^+
\end{aligned}
$$

by dominated convergence theorem. Thus $\{S_\rho(t)\ ;\ t\geqq 0\}$ is exponentially bounded C_ρ-semigroup with $\|S_\rho(t)\|\leqq Me^{(a+\varepsilon)t}$ for $0<\rho\leqq\rho_0$.

Next we shall prove that $A_\rho x=\rho^{-1}(F(\rho)x-x)$ is the C_ρ-c. i. g. of $\{S_\rho(t)\ ;\ t\geqq 0\}$ for $x\in X$. Let $T_\rho(t)x=C_\rho^{-1}S_\rho(t)x$ for $x\in D(T_\rho(t))=\{x\in X\ ;\ S_\rho(t)x\in R(C_\rho)\}$ and $t\geqq 0$. we define the operator G_ρ by $D(G_\rho)=\{x\in R(C_\rho)\ ;\ \lim_{t\to 0^+}\frac{T_\rho(t)x-x}{t}$ exists$\}$ and $G_\rho x=\lim_{t\to 0^+}\frac{T_\rho(t)x-x}{t}$ for $x\in D(G_\rho)$. Let $y\in R(C_\rho)$, $y=C_\rho x$, $x\in X$. By (ii)

$$
\begin{aligned}
&\left\|\frac{T_\rho(t)y-y}{t}-A_\rho y\right\|\\
&=\left\|\frac{1}{t}\left(e^{-\frac{t}{\rho}}\sum_{k=0}^{\infty}\left(\frac{t}{\rho}\right)^k\frac{F(\rho)^k}{k!}C_\rho-e^{-\frac{t}{\rho}}C_\rho\right)x\right.\\
&\qquad\left.+\frac{1}{t}(e^{-\frac{t}{\rho}}-I)C_\rho x-\frac{1}{\rho}(F(\rho)C_\rho-C_\rho)x\right\|\\
&=\left\|\frac{1}{t}\left(e^{-\frac{t}{\rho}}\sum_{k=1}^{\infty}\left(\frac{t}{\rho}\right)^k\frac{F(\rho)^k}{k!}C_\rho x\right)+\frac{1}{t}(e^{-\frac{t}{\rho}}-I)C_\rho x\right.\\
&\qquad\left.-\frac{1}{\rho}(F(\rho)C_\rho-C_\rho)x\right\|\\
&=\left\|\frac{1}{\rho}(e^{-\frac{t}{\rho}}-I)F(\rho)C_\rho x+\frac{1}{\rho}e^{-\frac{t}{\rho}}\sum_{k=2}^{\infty}\left(\frac{t}{\rho}\right)^{k-1}\frac{F(\rho)^k}{k!}C_\rho x\right.\\
&\qquad\left.+\left(\frac{e^{-\frac{t}{\rho}}-I}{t}+\frac{1}{\rho}\right)C_\rho x\right\|\\
&\leqq\frac{1}{\rho}\|e^{-\frac{t}{\rho}}-I\|\|F(\rho)C_\rho\|\|x\|+\frac{1}{\rho}e^{-\frac{t}{\rho}}\sum_{k=2}^{\infty}\left(\frac{t}{\rho}\right)^{k-1}\frac{\|F(\rho)^kC_\rho\|}{k!}\|x\|\\
&\qquad+\left\|\frac{e^{-\frac{t}{\rho}}-I}{t}+\frac{1}{\rho}\right\|\|C_\rho x\|\\
&\leqq\frac{1}{\rho}Me^{a\rho}\|e^{-\frac{t}{\rho}}-I\|\|x\|+\frac{M}{\rho}e^{-\frac{t}{\rho}+a\rho}\sum_{k=2}^{\infty}\left(\frac{t}{\rho}\right)^{k-1}\frac{e^{a\rho(k-1)}}{(k-1)!}\|x\|\\
&\qquad+\left\|\frac{e^{-\frac{t}{\rho}}-I}{t}+\frac{1}{\rho}\right\|\|C_\rho x\|
\end{aligned}
$$

$$=\frac{1}{\rho}M\,e^{a\rho}\|e^{-\frac{t}{\rho}}-I\|\|x\|+\frac{M}{\rho}e^{-\frac{t}{\rho}+a\rho}\sum_{k=1}^{\infty}\left(\frac{t}{\rho}\right)^{k}\frac{e^{a\rho k}}{k!}\|x\|$$

$$+\left\|\frac{e^{-\frac{t}{\rho}}-I}{t}+\frac{1}{\rho}\right\|\|C_\rho x\|$$

$$=\frac{M}{\rho}e^{a\rho}\|e^{-\frac{t}{\rho}}-I\|\|x\|+\frac{M}{\rho}e^{-\frac{t}{\rho}+a\rho}\left(\sum_{k=0}^{\infty}\left(\frac{t}{\rho}\right)^{k}\frac{e^{a\rho k}}{k!}-I\right)\|x\|$$

$$+\left\|\frac{e^{-\frac{t}{\rho}}-I}{t}+\frac{1}{\rho}\right\|\|C_\rho x\|$$

$$=\frac{1}{\rho}M\,e^{a\rho}\|e^{-\frac{t}{\rho}}-I\|\|x\|+\frac{M}{\rho}e^{-\frac{t}{\rho}+a\rho}\left[\exp\left(\frac{t}{\rho}e^{a\rho}\right)-I\right]\|x\|$$

$$+\left\|\frac{e^{-\frac{t}{\rho}}-I}{t}+\frac{1}{\rho}\right\|\|C_\rho x\|$$

$$\to 0 \text{ as } t\to 0^+$$

for $y\in R(C_\rho)$ and each fixed $\rho>0$.
Therefore $\lim_{t\to 0^+}\frac{T_\rho(t)x-x}{t}=A_\rho x(=G_\rho x)$ for $x\in R(C_\rho)$ and $t\geqq 0$.
Since A_ρ is closed operator, A_ρ is the C_ρ-c. i. g. of $\{S_\rho(t)\ ;\ t\geqq 0\}$ and that $\|S_\rho(t)\|\leqq Me^{(a+\varepsilon)t}$ for $0<\rho\leqq\rho_0$. From Theorem 2.5 and (i) it follows that A is closable and $\bar{A}$ is the C-c. i. g. of an exponentially bounded C-semigroup. If $\{S(t)\ ;\ t\geqq 0\}$ is an exponentially bounded C-semigroup generated by $\bar{A}$ then Theorem 2.5 implies that

(3.7) $$\|S_\rho(t)x-S(t)x\|\to 0 \text{ as } \rho\to 0$$

uniformly on bounded intervals and that $\|S(t)\|\leqq M\,e^{(a+\varepsilon)t}$ for $0<\rho\leqq\rho_0$. On the other hand, it follows from Lemma 3.1 that

$$\|S_{\rho_n}(\rho_n k_n)x-F(\rho_n)^{k_n}C_{\rho_n}x\|\leqq M\exp(a\rho_n(k_n-1)$$
$$+(e^{a\rho_n}-1)k_n)\{k_n^2(e^{a\rho_n}-1)^2+k_n e^{a\rho_n}\}^{\frac{1}{2}}\cdot\rho_n\left\|\frac{F(\rho_n)x-x}{\rho_n}\right\|$$

for $x\in X$. Choosing $x\in D$, $\rho_n\to 0$, $k_n\to\infty$ such that $\rho_n k_n\to t$ it is obvious that $\rho_n k_n$, $(e^{a\rho_n}-1)k_n$ and $\rho_n^{-1}\|F(\rho_n)x-x\|$ bounded as $n\to\infty$. Therefore we have

(3.8) $$\|S_{\rho_n}(\rho_n k_n)x-F(\rho_n)^{k_n}C_{\rho_n}x\|\leqq M'\rho_n^{\frac{1}{2}}\to 0 \text{ as } n\to\infty.$$

If $\rho_n=t/k_n$ we can choose the constants M' independent of t for $0\leqq t<T$, which implies, in this case, uniform convergence on bounded intervals in (3.7). For $x\in D$ we have

$$\|S(t)x-F(\rho_n)^{k_n}C_{\rho_n}x\|\leq\|S(t)x-S_{\rho_n}(t)x\|+\|S_{\rho_n}(t)x-S_{\rho_n}(k_n\rho_n)x\|$$
$$+\|S_{\rho_n}(k_n\rho_n)x-F(\rho_n)^{k_n}C_{\rho_n}x\|$$
$$=I_1+I_2+I_3.$$

Erom (3.7) and (3.8) it follows that $I_1\to 0$ and $I_3\to 0$ as $n\to\infty$. It is easy to see that

$$\frac{d}{ds}S_{\rho_n}(s)x=S_{\rho_n}(s)\frac{F(\rho_n)x-x}{\rho_n}\quad\text{for } x\in D.$$

Integrating this from $s=\rho_n k_n$ to $s=t$

$$S_{\rho_n}(t)x-S_{\rho_n}(\rho_n k_n)x=\int_{\rho_n k_n}^{t}S_{\rho_n}(s)\frac{F(\rho_n)x-x}{\rho_n}ds.$$

Since $\|S_{\rho_n}(t)\|\leq Me^{(a+\varepsilon)t}$ for large value of n, we have

$$I_2=\|S_{\rho_n}(t)x-S_{\rho_n}(\rho_n k_n)x\|\leq\int_{\rho_n k_n}^{t}\|S_{\rho_n}(s)\|\left\|\frac{F(\rho_n)x-x}{\rho_n}\right\|ds$$
$$\leq M\,e^{(a+\varepsilon)T}\|t-\rho_n k_n\|\left\|\frac{F(\rho_n)x-x}{\rho_n}\right\|$$
$$\to 0 \text{ as } n\to\infty$$

for $x\in D$ and $0\leq t\leq T$.

If $\rho_n=t/k_n$, then $I_2=0$. This concludes the proof of (3.6) for $x\in D$. Since D is dense in X and $\|S(t)-F(\rho_n)^{k_n}C_{\rho_n}\|$ is uniformly bounded, (3.6) holds for every $x\in X$. Finally the exponentially bounded C-semigroup generated by A satisfies $\|S(t)\|\leq M\,e^{(a+\varepsilon)t}$ for every small enough $\varepsilon>0$ and satisfies $\|S(t)\|\leq M\,e^{at}$.

Corollary 3.3. *Let $\{F(\rho)\,;\,\rho\geq 0\}$ satisfies (ii)$\sim$(iv) and let A be the C-c.i.g. of an exponentially C-semigroup $\{S(t)\,;\,t\geq 0\}$. We suppose that*

(a) *there exists an injective bounded linear operator C with dense range such that*

$$\lim_{\rho\to 0} C_\rho x=Cx \text{ for } x\in X,$$

(b) $\lim_{\rho\to 0}\rho^{-1}(F(\rho)x-x)=Ax$ *for* $x\in D(A)$.

Then

$$S(t)x=\lim_{n\to\infty}F\left(\frac{t}{n}\right)^{n}C_{\frac{t}{n}}x \text{ for } x\in X, \tag{3.9}$$

$\|S(t)\|\leq M\,e^{at}$ *and the limit in* (3.9) *is uniform for t bounded intervals on* $[0,\infty)$.

Proof. Since A is the C-c.i.g. of an exponentially bounded

C-semigroup it is closed, $D((\lambda-A)^{-k}\supset R(C)$ for $k=1,2,3,\cdots$, $\mathrm{Re}\,\lambda>a$ and $D(A)$ is dense in X. Therefore our result follows readily from Theorem 3.2.

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