SPACE-LIKE COMPLEX HYPERSURFACES OF
A COMPLEX LORENTZ MANIFOLD

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(Dedicated to Dr. T.Y. Rhee for his sixtieth birthday)

0. Introduction

It is recently proved by Aiyama and the authors [2] that a complete space-like complex submanifold of a complex space form \( M_{p+t}^{n+c'} \) \((c' \geq 0)\) is to totally geodesic. This is a complex version of the Bernstein-type theorem in the Minkowski space due to Calabi [4] and Cheng and Yau [5], which is generalized by Nishikawa [7] in the Lorentz manifold satisfying the strong energy condition. The purpose of this paper is to consider his result in the complex Lorentz manifold and is to prove the following

**Theorem.** Let \( M' \) be an indefinite Kaehler manifold of index 2 and let \( M \) be a complete space-like complex hypersurface of \( M' \). If \( M' \) is locally symmetric and if the horomorphic bisectional curvature for any space-like planes is non-negative and the holomorphic bisectional curvature for any space-like plane and any time-like plane is non-positive, then \( M \) is totally geodesic.

1. Indefinite complex submanifolds

This section is concerned with indefinite complex hypersurfaces of an indefinite Kaehler manifold. Let \((M', g')\) be an \((n+1)\)-dimensional connected indefinite Kaehler manifold of index \(2(s+t)\) \((0 \leq s \leq n, \ t = 0, 1)\) and let \( M \) be an \( n \)-dimensional complex hypersurface of index \( 2s \) of \( M' \). Then \( M \) is the indefinite Kaehler manifold endowed with the induced metric tensor \( g \). We choose a
local unitary frame field \( \{E_i\} = \{E_0, E_1, \cdots, E_n\} \) on a neighborhood of \( M' \) in such a way that, restricted to \( M \), \( E_1, \cdots, E_n \) are tangent to \( M \) and the other is normal to \( M \). Here and in the sequel the following convention on the range of indices are used throughout this paper, unless otherwise stated:

\[
A, B, \cdots = 0, 1, \cdots, n, \\
i, j, \cdots = 1, \cdots, n.
\]

With respect to the frame field, let \( \{\omega_i\} = \{\omega_0, \omega_i\} \) be its dual frame field. Then the Kaehler metric tensor \( g' \) of \( M' \) is given by

\[
g' = 2\Sigma \varepsilon_\alpha \omega_\alpha \otimes \bar{\omega}_\alpha.
\]

Associated with the frame field \( \{E_i\} \), there exist complex-valued 1-forms \( \omega_{AB} \), which are usually connection forms on \( M' \), such that they satisfy the structure equations of \( M' \):

\[
(1.1) \quad d\omega_A + \Sigma_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0,
\]

where \( \varepsilon_\alpha = \pm 1 \) and \( Q' = (Q'_{\alpha\beta}) \) (resp., \( R'_{\alpha\beta\gamma\delta} \)) denotes the curvature form (resp., the components of the indefinite Riemannian curvature tensor \( R' \)) of \( M' \).

Restricting these forms to the hypersurface \( M \), we have

\[
(1.2) \quad \omega_0 = 0
\]

and the induced indefinite Kaehler metric \( g \) of index 2s of \( M \) is given by

\[
g = 2\Sigma j \varepsilon_j \omega_j \otimes \bar{\omega}_j.
\]

Thus \( \{E_j\} \) is a local unitary frame field with respect to this metric and \( \{\omega\} \) is a local dual frame field due to \( \{E_j\} \), which consists of complex-valued 1-forms of type (1, 0) on \( M \). Moreover \( \omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n \) are linearly independent, and they are canonical forms on \( M \). It follows from (1.2) and the Cartan lemma that the exterior derivative of (1.2) gives rise to

\[
(1.3) \quad \omega_{ij} = \Sigma j \varepsilon_j h_{ij}, \quad h_{ij} = h_{ji}.
\]

The quadratic form \( \Sigma_i \varepsilon_i \varepsilon_j h_{ij} \omega_i \otimes \omega_j \otimes E_0 \) with values in the normal bundle is called the second fundamental form of the hypersurface \( M \), where we put \( \varepsilon = \varepsilon_0 \).

From the structure equations of \( M' \) it follows that the structure equations for \( M \) are similarly given by

\[
(1.4) \quad d\omega_i + \Sigma k \varepsilon_k \omega_{ik} \wedge \omega_k = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0,
\]

\[
d\omega_{ij} + \Sigma l \varepsilon_l \omega_{l(i} \wedge \omega_{j)} = Q_{ij}, \quad Q_{ij} = \Sigma k \varepsilon_k \varepsilon_j R_{ijki} \omega_k \wedge \bar{\omega}_i.
\]

Moreover the following relationships are defined:

\[
(1.5) \quad d\omega_{00} = Q_{00}, \quad Q_{00} = \Sigma k \varepsilon_k \varepsilon_l R_{00kl} \omega_k \wedge \bar{\omega}_l,
\]
where $\mathcal{O}_{g_0}$ is called the \textit{normal curvature form} of $M$. For the Riemannian curvature tensor $R$ and $R'$ of $M$ and $M'$ respectively, it follows from (1.1), (1.3) and (1.4) that we have the Gauss equation

\begin{equation}
R_{ijkl} = R'_{ijkl} - \varepsilon h_{jk} h_{il},
\end{equation}
and by means of (1.1), (1.3) and (1.5) we have

\begin{equation}
R_{g_{kl}} = R'_{g_{kl}} + \Sigma_{j} \varepsilon j h_{kj} h_{il}.
\end{equation}

The components $R_{ijkl}^m$ and $R_{ijkl}^m$ of the covariant derivative of the Riemannian curvature tensor $R$ are defined by

\begin{equation}
\Sigma_{m} \varepsilon_{m} (R_{ijkl}^m \omega_m + R_{ijkl}^m \omega_m) = dR_{ijkl} - \Sigma_{m} \varepsilon_{m} (R_{ijkl}^m \omega_m + R_{ijkl}^m \omega_m + \Gamma_{ijkl}^m \omega_m).
\end{equation}

The second Bianchi formula is given by

\begin{equation}
R_{ijkl}^m = R_{ijkl}^m.
\end{equation}

A plane section $P$ of the tangent space $T_x M$ of $M$ at any point $x$ is said to be \textit{non-degenerate}, provided that $g_x | T_x M$ is non-degenerate. It is easily seen that $P$ is non-degenerate if and only if it has a basis $\{u, v\}$ such that

\begin{equation}
g(u, u) g(v, v) - g(u, v)^2 \neq 0,
\end{equation}

and a holomorphic plane spanned by $u$ and $Ju$ is non-degenerate if and only if it contains some vector $v$ with $g(v, v) \neq 0$, where $J$ denotes the complex structure of $M$. The sectional curvature of the non-degenerate holomorphic plane $P$ spanned by $u$ and $Ju$ is called the \textit{holomorphic sectional curvature}, which is denoted by $H(P) = H(u)$. The indefinite Kaehler manifold $M$ is said to be \textit{of constant holomorphic sectional curvature} if its holomorphic sectional curvature $H(P)$ is constant for all $P$ and for all points of $M$. Then $M$ is called an \textit{indefinite complex space form}.

Now, the components $h_{ijk}$ and $h_{ijk}$ of the covariant derivative of the second fundamental form of $M$ are given by

\begin{equation}
\Sigma_{k} \varepsilon_{k} (h_{ijk} \omega_k + h_{ijk} \omega_k) = dh_{ij} - \Sigma_{k} \varepsilon_{k} (h_{ijk} \omega_k + h_{ijk} \omega_k) + \varepsilon h_{ijk} \omega_0.
\end{equation}

Then, substituting $dh_{ij}$ in this definition into the exterior derivative of (1.3), we have

\begin{equation}
h_{ijk} = h_{ikj} = h_{kij}, \quad h_{ijk} = - R'_{bij}.
\end{equation}

Similarly the components $h_{ijkl}$ and $h_{ijkl}$ (resp. $h_{ijkl}$ and $h_{ijkl}$) of the covariant derivative of $h_{ijk}$ (resp. $h_{ijk}$) can be defined by

\begin{equation}
\Sigma_{l} \varepsilon_{l} (h_{ijkl} \omega_l + h_{ijkl} \omega_l) = dh_{ijkl} - \Sigma_{l} \varepsilon_{l} (h_{ijkl} \omega_l + h_{ijkl} \omega_l + h_{ijkl} \omega_l + h_{ijkl} \omega_l) + \varepsilon h_{ijkl} \omega_0,
\end{equation}

\begin{equation}
\Sigma_{l} \varepsilon_{l} (h_{ijkl} \omega_l + h_{ijkl} \omega_l) = dh_{ijkl} - \Sigma_{l} \varepsilon_{l} (h_{ijkl} \omega_l + h_{ijkl} \omega_l + h_{ijkl} \omega_l + h_{ijkl} \omega_l) + \varepsilon h_{ijkl} \omega_0,
\end{equation}

\[ \ldots \text{7} \text{7} \ldots \]
and the simple calculation gives rise to
\begin{equation}
\begin{aligned}
\Omega_{ij} &= \Omega_{jk}, \\
\Omega_{ij} &= \Omega_{jk} - \sum_{\nu} \varepsilon_{\nu} (R_{\lambda ij\mu} \omega_{j\nu} + R_{\lambda jk\nu} h_{i\mu}) - \varepsilon R_{\lambda \mu \nu \lambda} h_{ij}.
\end{aligned}
\end{equation}

2. Proof of theorem

In order to prove the theorem, the Laplacian of the second fundamental form is first calculated. Let \( M' \) be an \((n+1)\)-dimensional indefinite locally symmetric Kaehler manifold and let \( M \) be a space-like complex hypersurface on \( M' \). From the second definition of \((1.10)\) together with the second equation of \((1.9)\) it follows that
\begin{equation}
\begin{aligned}
\sum_{\mu} \varepsilon_{\mu} \left( \Omega_{ij\mu} \omega_{k} + \Omega_{ij\mu} \omega_{k} \right)
- dR_{\lambda ij\mu} + \sum_{\alpha \mu} \varepsilon_{\alpha} (R'_{\lambda \alpha ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha} + R'_{\lambda ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha}) \\
+ \sum_{\alpha} \varepsilon_{\alpha} (R'_{\lambda \alpha ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha}) - \sum_{\alpha} \varepsilon_{\alpha} (R'_{\lambda \alpha ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha}) = \sum_{\alpha} \varepsilon_{\alpha} (R'_{\lambda \alpha ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha}) \omega_{k}, \\
+ \sum_{\alpha} \varepsilon_{\alpha} (R'_{\lambda \alpha ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha}) = \sum_{\alpha} \varepsilon_{\alpha} (R'_{\lambda \alpha ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha}) \omega_{k},
\end{aligned}
\end{equation}

from which it turns out that
\begin{equation}
\begin{aligned}
\Omega_{ij} &= -\varepsilon (R'_{\lambda \alpha ij} \omega_{\mu} + R'_{\lambda \mu ij} \omega_{\alpha}) + \sum_{\alpha} \varepsilon_{\alpha} R'_{\alpha ij} \omega_{k},
\end{aligned}
\end{equation}

because the ambient space \( M' \) is locally symmetric. Therefore, by means of \((1.11)\), the covariant derivative \( \Omega_{ij\lambda} \) of \( \Omega_{ij} \) is obtained:
\begin{equation}
\begin{aligned}
\Omega_{ij\lambda} &= \sum_{\mu} \{ (R'_{\lambda \alpha ij} - \varepsilon \varphi_{ij} \varphi_{\alpha \lambda}) \omega_{\mu} + (R'_{\lambda \alpha ij} - \varepsilon \varphi_{ij} \varphi_{\lambda \alpha}) \omega_{\mu} \} \\
- \varepsilon (R'_{\lambda \alpha ij} + \sum_{\alpha} \varepsilon_{\alpha} \varphi_{\mu \alpha} \varphi_{\lambda \alpha} \omega_{\mu} + \varepsilon (R'_{\lambda \alpha ij} + R'_{\lambda \mu ij} \omega_{\lambda}) \\
+ \sum_{\alpha} \varepsilon_{\alpha} R'_{\alpha ij} \omega_{k},
\end{aligned}
\end{equation}

from which it follows that
\begin{equation}
\begin{aligned}
\sum_{\mu} \varepsilon_{\mu} \{ \sum_{\alpha} (R'_{\alpha \mu ij} \omega_{\nu} + R'_{\alpha \mu ij} \omega_{\nu}) - \{ (\varphi_{ij})^2 \omega_{\nu} + (\varphi_{ij})^2 \omega_{\nu} \} \} \\
- \sum_{\alpha} \varepsilon (R'_{\alpha \mu ij} \omega_{\nu} - \varphi_{ij} \varphi_{\nu} \omega_{\mu} + \varepsilon (R'_{\alpha \mu ij} \omega_{\nu} + R'_{\alpha \mu ij} \omega_{\nu}) + \varepsilon (R'_{\alpha \mu ij} \omega_{\nu} + R'_{\alpha \mu ij} \omega_{\nu}) \omega_{k},
\end{aligned}
\end{equation}

where \((\varphi_{ij})^2 \Sigma_{\alpha} \varepsilon_{\alpha} \varphi_{ij} \varphi_{\alpha \lambda} \varphi_{ji} \varphi_{\lambda \alpha} \omega_{\mu} + \varepsilon (R'_{\alpha \mu ij} \omega_{\nu} + R'_{\alpha \mu ij} \omega_{\nu}) \omega_{k},
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\frac{1}{2} \Delta h_{ij} &= \frac{1}{2} \sum_{\alpha} \varepsilon_{\alpha} (R'_{\alpha \mu ij} \omega_{\nu} + R'_{\alpha \mu ij} \omega_{\nu}) \omega_{k} + 2 \sum_{\alpha} \varepsilon_{\alpha} R'_{\alpha \mu ij} (h_{ij})^2 \\
- \sum_{\alpha} \varepsilon_{\alpha} R'_{\alpha \mu ij} (h_{ij}^2) - 2 h_{ij} - h_{ij}^2 - 4 \sum_{\alpha} \varepsilon (R'_{\alpha \mu ij} (h_{ij})^2 + 2 \sum_{\alpha} \varepsilon_{\alpha} \varphi_{ij} \varphi_{\nu} \omega_{\mu} + \varepsilon (R'_{\alpha \mu ij} \omega_{\nu} + R'_{\alpha \mu ij} \omega_{\nu}) \omega_{k},
\end{aligned}
\end{equation}

where \((h_{ij})^2 \Sigma_{\alpha} \varepsilon_{\alpha} \varphi_{ij} \varphi_{\alpha \lambda} \varphi_{ji} \varphi_{\lambda \alpha} \omega_{\mu} + \varepsilon (R'_{\alpha \mu ij} \omega_{\nu} + R'_{\alpha \mu ij} \omega_{\nu}) \omega_{k},
\end{aligned}
\end{equation}

for the holomorphic bisectional curvatures of the ambient space \( M' \), the following conditions are assumed: There exist constants \( c_{1} \) and \( c_{2} \) such that
(a) $H'(P_1', P_2') \geq c_1$ for any space-like $J$-invariant planes $P_1'$ and $P_2'$, and $P_1' \perp P_2'$:

(b) $H'(P', Q') \leq c_2$ for any space-like $J$-invariant plane $P'$ and any time-like $J$-invariant plane $Q'$.

Since $M$ is space-like, the matrix $((h_{jk})^2)$ is a negative semi-definite Hermitian one and it is diagonalizable. Then a local field of unitary frames on $M$ can be chosen in such a way that $(h_{jk})^2 = \lambda_j \delta_{jk}$, where the eigenvalues $\lambda_j$'s are non-positive real valued functions on $M$. Thus we have

$$h_{ij} = \Sigma_j \lambda_j^2 \geq (-h_2)^2/n.$$

On the other hand, $(h_{ij} \bar{h}_{kl})$ can be regarded as a Hermitian matrix of order $n^2$ which is positive semi-definite and it is also diagonalizable. Accordingly, it has eigenvalues $\lambda_{ij}^r$ which are non-negative and we can express as $(h_{ij} \bar{h}_{kl}) = \lambda_{ij} \delta_{kl}$. Thus, by definition, we have

$$\Sigma_i \lambda_{ij} = -h_2,$$

and hence

$$\Sigma_{i,j,k,l} R'_{ikkl} h_{ij} \geq c_1 \Sigma_i \lambda_{ij} = -c_1 h_2,$$

because of the condition (a). From these inequalities it follows that the equation (2.2) is deformed onto

(2.3) \hspace{2cm} \Delta f \geq 2 \{2(n+1)c_1 - (n+4)c_2 \} f + 2(n+2)f^2/n \hspace{2cm}

for a non-negative function $f$ defined by $-h_2$, where the equality holds true if and only if $\lambda_j = \lambda$ for any indices $j$ and $h_{jk}$ are parallel. Under these preparations, the following Proposition is proved.

**Proposition 2.1.** Let $M'$ be an $(n+1)$-dimensional indefinite Kaehler manifold of index 2 and let $M$ be a complete space-like complex hypersurface of $M'$. If $M'$ is locally symmetric and satisfies the conditions (a) and (b) such that $[2(n+1)c_1 - (n+4)c_2) \geq 0$, then $M$ is totally geodesic.

**Remark.** Let $M$ be an $n(\geq 2)$-dimensional indefinite Kaehler manifold. In [3], Barros and Romero proved that if $M$ has the bisectional curvature bounded from above and bounded from below, then $M$ is an indefinite complex space form. Although this fact means that the ambient space $M'$ in proposition 2.1 is very close to an indefinite complex space form, they are not the same ones. In fact, there exists such an indefinite Kaehler manifold in Propo-
sition 2.1 which is not of constant holomorphic curvature. This example is next given.

**Example.** For the complex coordinate system \( \{ z_i, z_{n+j}, z_{2n+1} \} \) of a 
\((2n+1)\)-dimensional complex Euclidean space \( \mathbb{C}_i^{2n+1} \) of index \( 2s \), let 
\( M' \) be an indefinite complex hypersurface of \( \mathbb{C}_i^{2n+1} \) given by the 
equation
\[
z_{2n+1} = \sum_{j=1}^{n} (z_j + c_j z_{n+j})^2, \quad c_j \in \mathbb{C}, \quad |c_j| = 1.
\]
Then it is seen in [1] and [9] that \( M' \) is a complete complex 
hypersurface of index \( 2s \) of \( \mathbb{C}_i^{2n+1} \) which are locally symmetric, but 
not flat. The straightforward calculation implies that \( M' \) satisfies 
the conditions (a) and (b), \( c_1 = c_2 = -4 \). However, \( M' \) is not a 
complex space form, because it is not flat.

In order to prove Proposition 2.1, the following theorem due to 
Omori [8] for the estimate of the Laplacian of the function of 
class \( C^2 \) is needed. The original one is slightly different from that 
quoted here, which is used by Ishihara [6] to prove the Bernstein-
type theorem of complete space-like submanifold of an anti-de 
Sitter space.

**Theorem (Omori).** Let \( N \) be a complete Riemannian manifold 
whose Ricci curvature is bounded below and let \( F \) be a function of 
class \( C^2 \) on \( N \). If \( F \) is bounded below, then for any point \( p \) and 
\( \varepsilon > 0 \) there exists a point \( q \) depending on \( p \) such that 
\[
(2.4) \quad |\text{grad} \, F(q)| < \varepsilon, \quad \Delta F(q) > -\varepsilon, \quad F(q) \leq F(p).
\]

**Proof of Proposition 2.1.** Since the right hand side of (2.3) is 
non-negative, the conclusion is trivial by the maximal principal if 
\( M \) is compact.

Now, \( M \) is assumed to be non-compact and complete. For any 
positive number \( a \), a function \( F \) on \( M \) defined by 
\[ \frac{1}{2} (f + a)^{1/2} \] is smooth and bounded. Since the Ricci curvature of \( M \) is bounded 
from below by \( nc_1 \) because of (1.6), and the function \( F \) is also 
bounded below, the theorem due to Omori can applied to \( F \). It 
means that for any point \( p \) and for any \( \varepsilon > 0 \) there exists a point 
\( q \) at which \( F \) satisfies (2.4), from which it follows that 
\[
2\varepsilon (3\varepsilon + F(q)) > F(q)^2 \Delta f(q) \geq 0
\]
by the direct calculation. When \( \varepsilon \) tends to 0, the left hand side
converges to 0, because the function $F$ is bounded. For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ ($m \to \infty$), there exists a point sequence $\{q_m\}$ so that the sequence $\varepsilon_m(3\varepsilon_m + F(q_m))$ converges to 0 by taking a subsequence if necessary, and hence we have

$$F(q_m)^4 \Delta f(q_m) \to 0.$$  

By (2.3) and the definition of $F$ it yields that

$$(2.5) \quad \{(n+2)f(q_m)^2 + n\{2(n+1)c_1 + (n+4)c_2\}f(q_m)\}$$

$$/\{f(q_m) + a\}^2 \to 0,$$

from which it follows that, given an arbitrary small $\varepsilon > 0$, there exists a positive integer $N$ such that

$$\{(n+2) - \varepsilon\}f(q_m)^2 + [n\{2(n+1)c_1 + (n+4)c_2\} - 2ae]f(q_m) - a^2\varepsilon < 0$$

for any $m > N$. Since this means that the sequence $\{f(q_m)\}$ is bounded, there is a subsequence of $\{f(q_m)\}$ which converges to $f_0$ ($\geq 0$). Combining this fact together with (2.5), we have $f_0 = 0$ and $\{F(q_m)\}$ converges to $a^{-1/2}$. Since we have $F(q_m) \leq F(p)$ at any fixed point $p$ because of (2.4), the point is the maximal one of $F$ and therefore $f(p) = -h_2 = 0$. This shows that $M$ is totally geodesic, because $M$ is space-like.

The theorem in the introduction is an immediate consequence of the Proposition 2.1.

Bibliography

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