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# SPACE-LIKE COMPLEX HYPERSURFACES OF A COMPLEX LORENTZ MANIFOLD

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(Dedicated to Dr. T. Y. Rhee for his sixtieth birthday)

## 0. Introduction

It is recently proved by Aiyama and the authors [2] that a complete space-like complex submanifold of a complex space form $M_p^{n+p}(c')$ $(c' \geqq 0)$ is to totally geodesic. This is a complex version of the Bernstein-type theorem in the Minkowski space due to Calabi [4] and Cheng and Yau [5], which is generalized by Nishikawa [7] in the Lorentz manifold satisfying the strong energy condition. The purpose of this paper is to consider his result in the complex Lorentz manifold and is to prove the following

THEOREM. *Let $M'$ be an indefinite Kaehler manifold of index 2 and let $M$ be a complete space-like complex hypersurface of $M'$. If $M'$ is locally symmetric and if the horomorphic bisectional curvature for any space-like planes is non-negative and the holomorphic bisectional curvature for any space-like plane and any time-like plane is non-positive, then $M$ is totally geodesic.*

## 1. Indefinite complex submanifolds

This section is concerned with indefinite complex hypersurfaces of an indefinite Kaehler manifold. Let $(M', g')$ be an $(n+1)$-dimensional connected indefinite Kaehler manifold of index $2(s+t)$ $(0 \leqq s \leqq n,\ t=0, 1)$ and let $M$ be an $n$-dimensional complex hypersurface of index $2s$ of $M'$. Then $M$ is the indefinite Kaehler manifold endowed with the induced metric tensor $g$. We choose a

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local unitary frame field $\{E_A\}=\{E_0, E_1, \cdots, E_n\}$ on a neighborhood of $M'$ in such a way that, restricted to $M$, $E_1, \cdots, E_n$ are tangent to $M$ and the other is normal to $M$. Here and in the sequel the following convention on the range of indices are used throughout this paper, unless otherwise stated:

$$A, B, \cdots=0, 1, \cdots, n,$$
$$i, j, \cdots=1, \cdots, n.$$

With respect to the frame field, let $\{\omega_A\}=\{\omega_0, \omega_i\}$ be its dual frame field. Then the Kaehler metric tensor $g'$ of $M'$ is given by $g'=2\Sigma_A\varepsilon_A\omega_A\otimes\bar{\omega}_A$. Associated with the frame field $\{E_A\}$, there exist complex-valued 1-forms $\omega_{AB}$, which are usually connection forms on $M'$, such that they satisfy the structure equations of $M'$:

$$(1.1)\qquad d\omega_A+\Sigma_B\varepsilon_B\omega_{AB}\wedge\omega_B=0, \qquad \omega_{AB}+\bar{\omega}_{BA}=0,$$
$$d\omega_{AB}+\Sigma_C\varepsilon_C\omega_{AC}\wedge\omega_{CB}=\Omega'_{AB}, \quad \Omega'_{AB}=\Sigma_{C,D}\varepsilon_C\varepsilon_D R'_{\bar{A}BC\bar{D}}\omega_C\wedge\bar{\omega}_D,$$

where $\varepsilon_A=\pm 1$ and $\Omega'=(\Omega'_{AB})$ (resp, $R'_{\bar{A}BC\bar{D}}$) denotes the curvature form (resp. the components of the indefinite Riemannian curvature tensor $R'$) of $M'$.

Restricting these forms to the hypersurface $M$, we have

$$(1.2)\qquad \omega_0=0$$

and the induced indefinite Kaehler metric $g$ of index 2s of $M$ is given by $g=2\Sigma_j\varepsilon_j\omega_j\otimes\bar{\omega}_j$. Thus $\{E_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual frame field due to $\{E_j\}$, which consists of complex-valued 1-forms of type (1, 0) on $M$. Moreover $\omega_1, \cdots, \omega_n$, $\bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent, and they are cannonical forms on $M$. It follows from (1.2) and the Cartan lemma that the exterior derivative of (1.2) gives rise to

$$(1.3)\qquad \omega_{0i}=\Sigma_j\varepsilon_j h_{ij}\omega_j, \quad h_{ij}=h_{ji}.$$

The quadratic form $\Sigma_{i,j}\varepsilon\varepsilon_i\varepsilon_j h_{ij}\omega_i\otimes\omega_j\otimes E_0$ with values in the normal bundle is called the *second fundamental form* of the hypersurface $M$, where we put $\varepsilon=\varepsilon_0$.

From the structure equations of $M'$ it follows that the structure equations for $M$ are similarly given by

$$(1.4)\qquad d\omega_i+\Sigma_k\varepsilon_k\omega_{ik}\wedge\omega_k=0, \qquad \omega_{ij}+\bar{\omega}_{ji}=0,$$
$$d\omega_{ij}+\Sigma_k\varepsilon_k\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}, \quad \Omega_{ij}=\Sigma_{k,l}\varepsilon_k\varepsilon_l R_{\bar{i}jk\bar{l}}\omega_k\wedge\bar{\omega}_l.$$

Moreover the following relationships are defined:

$$(1.5)\qquad d\omega_{00}=\Omega_{00}, \quad \Omega_{00}=\Sigma_{k,l}\varepsilon_k\varepsilon_l R_{\bar{0}0k\bar{l}}\omega_k\wedge\bar{\omega}_l,$$

where $\Omega_{00}$ is called the *normal curvature form* of $M$. For the Riemannian curvature tensor $R$ and $R'$ of $M$ and $M'$ respectively, it follows from (1.1), (1.3) and (1.4) that we have the Gauss equation

(1.6) $$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} - \varepsilon h_{jk}\bar{h}_{il},$$

and by means of (1.1), (1.3) and (1.5) we have

(1.7) $$R_{\bar{0}0k\bar{l}} = R'_{\bar{0}0k\bar{l}} + \Sigma_j \varepsilon_j h_{kj}\bar{h}_{jl}.$$

The components $R_{\bar{i}jk\bar{l}m}$ and $R_{\bar{i}jk\bar{l}\bar{m}}$ of the covariant derivative of the Riemannian curvature tensor $R$ are defined by

$$\Sigma_m \varepsilon_m (R_{\bar{i}jk\bar{l}m}\omega_m + R_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) = dR_{\bar{i}jk\bar{l}} - \Sigma_m \varepsilon_m (R_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + R_{\bar{i}mk\bar{l}}\omega_{mj} + R_{\bar{i}jm\bar{l}}\omega_{mk} + R_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}).$$

The second Bianchi formula is given by

(1.8) $$R_{\bar{i}jk\bar{l}m} = R_{\bar{i}jm\bar{l}k}.$$

A plane section $P$ of the tangent space $T_xM$ of $M$ at any point $x$ is said to be *non-degenerate*, provided that $g_x | T_xM$ is non-degenerate. It is easily seen that $P$ is non-degenerate if and only if it has a basis $\{u, v\}$ such that

$$g(u,u)g(v,v) - g(u,v)^2 \neq 0,$$

and a holomorphic plane spanned by $u$ and $Ju$ is non-degenerate if and only if it contains some vector $v$ with $g(v,v) \neq 0$, where $J$ denotes the complex structure of $M$. The sectional curvature of the non-degenerate holomorphic plane $P$ spanned by $u$ and $Ju$ is called the *holomorphic sectional curvature*, which is denoted by $H(P) = H(u)$. The indefinite Kaehler manifold $M$ is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvature $H(P)$ is constant for all $P$ and for all points of $M$. Then $M$ is called an *indefinite complex space form*.

Now, the components $h_{ijk}$ and $h_{ij\bar{k}}$ of the covariant derivative of the second fundamental form of $M$ are given by

$$\Sigma_k \varepsilon_k (h_{ijk}\omega_k + h_{ij\bar{k}}\bar{\omega}_k) = dh_{ij} - \Sigma_k \varepsilon_k (h_{kj}\omega_{ki} + h_{ik}\omega_{kj}) + \varepsilon h_{ij}\omega_{00}.$$

Then, substituting $dh_{ij}$ in this definition into the exterior derivative of (1.3), we have

(1.9) $$h_{ijk} = h_{ikj} = h_{jik}, \quad h_{ij\bar{k}} = -R'_{\bar{0}ij\bar{k}}.$$

Similarly the components $h_{ijkl}$ and $h_{ijk\bar{l}}$ (resp. $h_{ij\bar{k}l}$ and $h_{ij\bar{k}\bar{l}}$) of the covariant derivative of $h_{ijk}$ (resp. $h_{ij\bar{k}}$) can be defined by

(1.10) $$\Sigma_l \varepsilon_l (h_{ijkl}\omega_l + h_{ijk\bar{l}}\bar{\omega}_l) = dh_{ijk} - \Sigma_l \varepsilon_l (h_{ljk}\omega_{li} + h_{ilk}\omega_{lj} + h_{ijl}\omega_{lk}) + \varepsilon h_{ijk}\omega_{00},$$
$$\Sigma_l \varepsilon_l (h_{ij\bar{k}l}\omega_l + h_{ij\bar{k}\bar{l}}\bar{\omega}_l) = dh_{ij\bar{k}} - \Sigma_l \varepsilon_l (h_{lj\bar{k}}\omega_{li} + h_{il\bar{k}}\omega_{lj} + h_{ij\bar{l}}\bar{\omega}_{lk}) + \varepsilon h_{ij\bar{k}}\omega_{00},$$

and the simple calculation gives rise to

(1. 11) $$h_{ijkl}=h_{ijlk},$$
$$h_{ijk\bar{l}}-h_{ij\bar{l}k}=\Sigma_m\varepsilon_m(R_{\bar{l}ki\bar{m}}h_{mj}+R_{\bar{l}kj\bar{m}}h_{im})-\varepsilon R_{\bar{0}0k\bar{l}}h_{ij}.$$

## 2. Proof of theorem

In order to prove theorem, the Laplacian of the second fundamental form is first calculated. Let $M'$ be an $(n+1)$-dimensional indefinite locally symmetric Kaehler manifold and let $M$ be a space-like complex hypersurface on $M'$. From the second definition of (1. 10) together with the second equation of (1. 9) it follows that

$$\Sigma_k\varepsilon_k(h_{ij\bar{l}k}\omega_k+h_{ij\bar{l}\bar{k}}\bar{\omega}_k)$$
$$=-dR'_{\bar{0}ij\bar{l}}-\Sigma_m\varepsilon_m(h_{mj\bar{l}}\omega_{mi}+h_{im\bar{l}}\omega_{mj}+h_{ij\bar{m}}\bar{\omega}_{ml})-\varepsilon R'_{\bar{0}ij\bar{l}}\omega_{00}$$
$$=-dR'_{\bar{0}ij\bar{l}}+\Sigma_A\varepsilon_A(R'_{\bar{0}Aj\bar{l}}\omega_{Ai}+R'_{\bar{0}iA\bar{l}}\omega_{Aj}+R'_{\bar{0}ij\bar{A}}\bar{\omega}_{Al})$$
$$+\Sigma_A\varepsilon_AR'_{\bar{A}ij\bar{l}}\bar{\omega}_{A0}-\Sigma_k\varepsilon\varepsilon_k(R'_{\bar{0}0j\bar{l}}h_{ik}\omega_k+R'_{\bar{0}0i\bar{l}}h_{jk}\omega_k+R'_{\bar{0}ij0}\bar{h}_{lk}\bar{\omega}_k)$$
$$+\Sigma_{m,k}\varepsilon_m\varepsilon_kR'_{\bar{m}ij\bar{l}}h_{mk}\omega_k$$
$$=-\Sigma_k\varepsilon_k(R'_{\bar{0}ij\bar{l}k}\omega_k+R'_{\bar{0}ij\bar{l}\bar{k}}\bar{\omega}_k)-\Sigma_k\varepsilon\varepsilon_k(R'_{\bar{0}0j\bar{l}}h_{ik}+R'_{\bar{0}0i\bar{l}}h_{jk})\omega_k$$
$$+\Sigma_{m,k}\varepsilon_m\varepsilon_kR'_{\bar{m}ij\bar{l}}h_{mk}\omega_k-\Sigma_k\varepsilon\varepsilon_kR'_{\bar{0}ij\bar{0}}\bar{h}_{lk}\bar{\omega}_k,$$

from which it turns out that

(2. 1) $$h_{ij\bar{l}k}=-\varepsilon(R'_{\bar{0}0j\bar{l}}h_{ik}+R'_{\bar{0}0i\bar{l}}h_{jk})+\Sigma_m\varepsilon_mR'_{\bar{m}ij\bar{l}}h_{mk},$$

because the ambient space $M'$ is locally symmetric. Therefore, by means of (1. 11), the covariant derivative $h_{ijk\bar{l}}$ of $h_{ijk}$ is obtained:

$$h_{ijk\bar{l}}=\Sigma_m\{(R'_{\bar{l}ki\bar{m}}-\varepsilon h_{ki}\bar{h}_{ml})h_{mj}+(R'_{\bar{l}kj\bar{m}}-\varepsilon h_{kj}\bar{h}_{ml})h_{mi}\}$$
$$-\varepsilon(R'_{\bar{0}0k\bar{l}}+\Sigma_m\bar{h}_{km}h_{ml})h_{ij}-\varepsilon(R'_{\bar{0}0j\bar{l}}h_{ik}+R'_{\bar{0}0i\bar{l}}h_{jk})$$
$$+\Sigma_mR'_{\bar{m}ij\bar{l}}h_{mk},$$

from which it follows that

$$\Sigma_kh_{ijk\bar{k}}=\Sigma_m[\Sigma_k(R'_{\bar{k}ki\bar{m}}h_{mj}+R'_{\bar{k}kj\bar{m}}h_{mi})-\{(h_{i\bar{m}})^2h_{mj}+(h_{j\bar{m}})^2h_{mi}\}]$$
$$-\Sigma_k\varepsilon R'_{00k\bar{k}}h_{ij}-h_2h_{ij}-\Sigma_k(R'_{\bar{0}0j\bar{k}}h_{ik}+R'_{\bar{0}0i\bar{k}}h_{jk})+\Sigma_{k,l}R'_{\bar{k}ij\bar{l}}h_{kl},$$

where $(h_{i\bar{j}})^2=\Sigma_k\varepsilon h_{ik}\bar{h}_{kj}$ and $h_2=\Sigma_j(h_{j\bar{j}})^2$. Taking account of (2. 1) and the above equation, we then have

(2. 2) $$\frac{1}{2}\Delta h_2=\Sigma_{i,j,k}\varepsilon(h_{ij\bar{k}}\bar{h}_{ij\bar{k}}+h_{ijk}\bar{h}_{ijk})+2\Sigma_{i,j,k}R'_{\bar{k}ki\bar{j}}(h_{j\bar{i}})^2$$
$$-\Sigma_k\varepsilon h_2R'_{\bar{0}0k\bar{k}}-2h_4-h_2^2-4\Sigma_{j,k}\varepsilon R'_{\bar{0}0j\bar{k}}(h_{\bar{j}k})^2$$
$$+2\Sigma_{i,j,k}\varepsilon R'_{\bar{k}ij\bar{l}}h_{kl}\bar{h}_{ij},$$

where $h_4=\Sigma_{j,k}(h_{j\bar{k}})^2(h_{\bar{j}k})^2$. For the holomorphic bisectional curvatures of the ambient space $M'$, the following conditions are assumed: There exist constants $c_1$ and $c_2$ such that

(a) $H'(P_1', P_2') \geqq c_1$ for any space-like $J$-invariant planes $P_1'$ and $P_2'$, and $P_1' \perp P_2'$:

(b) $H'(P', Q') \leqq c_2$ for any space-like $J$-invariant plane $P'$ and any time-like $J$-invariant plane $Q'$.

Since $M$ is space-like, the matrix $((h_{j\bar{k}})^2)$ is a negative semi-definite Hermitian one and it is diagonalizable. Then a local field of unitary frames on $M$ can be chosen in such a way that $(h_{j\bar{k}})^2 = \lambda_j \delta_{jk}$, where the eigenvalues $\lambda_j$'s are non-positive real valued functions on $M$. Thus we have

$$h_4 = \Sigma_j \lambda_j^2 \geqq (-h_2)^2/n.$$

On the other hand, $(h_{ij}\bar{h}_{kl})$ can be regarded as a Hermitian matrix of order $n^2$ which is positive semi-definite and it is also diagonalizable. Accordingly, it has eigenvalues $\lambda_{ij}$ which are non-negative and we can express as $(h_{ij}\bar{h}_{kl}) = \lambda_{ij}\delta^{ij}_{kl}$. Thus, by definition, we have

$$\Sigma_{i,j}\lambda_{ij} = -h_2,$$

and hence

$$\Sigma_{i,j,k,l} R'_{\bar{k}ij\bar{l}} \bar{h}_{kl} h_{ij} \geqq c_1 \Sigma_{i,j}\lambda_{ij} = -c_1 h_2,$$

because of the condition (a). From these inequalities it follows that the equation (2.2) is deformed onto

$$\Delta f \geqq 2\{2(n+1)c_1 - (n+4)c_2\} f + 2(n+2) f^2/n \tag{2.3}$$

for a non-negative function $f$ defined by $-h_2$, where the equality holds true if and only if $\lambda_j = \lambda$ for any indices $j$ and $h_{jk}$ are parallel. Under these preparations, the following Proposition is proved.

PROPOSITION 2.1. *Let $M'$ be an $(n+1)$-dimensional indefinite Kaehler manifold of index 2 and let $M$ be a complete space-like complex hypersurface of $M'$. If $M'$ is locally symmetric and satisfies the conditions (a) and (b) such that $[2(n+1)c_1 - (n+4)c_2] \geqq 0$, then $M$ is totally geodesic.*

REMARK. Let $M$ be an $n(\geqq 2)$-dimensional indefinite Kaehler manifold. In [3], Barros and Romero proved that if $M$ has the bisectional curvature bounded from above and bounded from below, then $M$ is an indefinite complex space form. Although this fact means that the ambient space $M'$ in proposition 2.1 is very close to an indefinite complex space form, they are not the same ones. In fact, there exists such an indefinite Kaehler manifold in Proposition 2.1 which is not of constant holomorphic curvature. This example is next given.

EXAMPLE. For the complex coordinate system $\{z_j, z_{n+j}, z_{2n+1}\}$ of a $(2n+1)$-dimensional complex Euclidean space $C_s^{2n+1}$ of index 2s, let $M'$ be an indefinite complex hypersurface of $C_s^{2n+1}$ given by the equation

$$z_{2n+1}=\Sigma_j(z_j+c_jz_{n+j})^2,\ \ c_j\in C,\ \ |c_j|=1.$$

Then it is seen in [1] and [9] that $M'$ is a complete complex hypersurface of index 2s of $C_s^{2n+1}$ which are locally symmetric, but not flat. The straightforward calculation implies that $M'$ satisfies the conditions (a) and (b), $c_1=c_2=-4$. However, $M'$ is not a complex space form, because it is not flat.

In order to prove Proposition 2.1, the following theorem due to Omori [8] for the estimate of the Laplacian of the function of class $C^2$ is needed. The original one is slightly different from that quoted here, which is used by Ishihara [6] to prove the Bernstein-type theorem of complete space-like submanifold of an anti-de Sitter space.

THEOREM (OMORI). *Let $N$ be a complete Riemannian manifold whose Ricci curvature is bounded below and let $F$ be a function of class $C^2$ on $N$. If $F$ is bounded below, then for any point $p$ and $\varepsilon>0$ there exists a point $q$ depending on $p$ such that*

$$|grad\, F(q)|<\varepsilon,\ \ \Delta F(q)>-\varepsilon,\ \ F(q)\leqq F(p). \tag{2.4}$$

*Proof of Proposition* 2.1. Since the right hand side of (2.3) is non-negative, the conclusion is trivial by the maximal principal if $M$ is compact.

Now, $M$ is assumed to be non-compact and complete. For any positive number $a$, a function $F$ on $M$ defined by $1/(f+a)^{1/2}$ is smooth and bounded. Since the Ricci curvature of $M$ is bounded from below by $nc_1$ because of (1.6), and the function $F$ is also bounded below, the theorem due to Omori can applied to $F$. It means that for any point $p$ and for any $\varepsilon>0$ there exists a point $q$ at which $F$ satisfies (2.4), from which it follows that

$$2\varepsilon\{3\varepsilon+F(q)\}>F(q)^4\Delta f(q)\geqq 0$$

by the direct calculation. When $\varepsilon$ tends to 0, the left hand side converges to 0, because the function $F$ is bounded. For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ $(m \to \infty)$, there exists a point sequence $\{q_m\}$ so that the sequence $\{\varepsilon_m(3\varepsilon_m + F(q_m)\}$ converges to 0 by taking a subsequence if necessary, and hence we have

$$F(q_m)^4 \Delta f(q_m) \to 0.$$

By (2.3) and the definition of $F$ it yields that

$$(2.5) \qquad \{(n+2)f(q_m)^2 + n\{2(n+1)c_1 + (n+4)c_2\}f(q_m)\} / \{f(q_m)+a\}^2 \to 0,$$

from which it follows that, given an arbitrary small $\varepsilon > 0$, there exists a positive integer $N$ such that

$$\{(n+2)-\varepsilon\}f(q_m)^2 + [n\{2(n+1)c_1 + (n+4)c_2\} - 2a\varepsilon]f(q_m) - a^2\varepsilon < 0$$

for any $m > N$. Since this means that the sequence $\{f(q^m)\}$ is bounded, there is a subsequence of $\{f(q_m)\}$ which converges to $f_0$ $(\geqq 0)$. Combining this fact together with (2.5), we have $f_0 = 0$ and $\{F(q_m)\}$ converges to $a^{-1/2}$. Since we have $F(q_m) \leqq F(p)$ at any fixed point $p$ because of (2.4), the point is the maximal one of $F$ and therefore $f(p) = -h_2 = 0$. This shows that $M$ is totally goedesic, because $M$ is space-like.

The theorem in the introduction is an immediate consequence of the Proposition 2.1.

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