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RELATIVELY OPEN SPECTRUM IN $L(X)$

Woo Young Lee

1. Introduction

Suppose X and Y are nonzero complex normed spaces, write $L(X, Y)$ for the bounded linear operators from X to Y, write X^* for the dual space of X, and if $T \in L(X, Y)$ write $T^* : Y^* \to X^*$ for the dual or *adjoint* of T. $L(X, Y)$ is abbreviated to $L(X)$. Recall, following [1], [3] or [4], that $T \in L(X, Y)$ is said to be *bounded below* if there is $k > 0$ for which

(1.1) $\qquad \|x\| \leq k\|Tx\| \qquad$ for all $x \in X$,

is said to be *open* if there is $k > 0$ for which

(1.2) $\qquad y \in \{Tx : \|x\| \leq k\|y\|\} \qquad$ for all $y \in Y$,

is said to be *almost open* if there is $k > 0$ for which

(1.3) $\qquad y \in \mathrm{cl}\{Tx : \|x\| \leq k\|y\|\} \qquad$ for all $y \in Y$,

and is said to be dense if T has a dense range.

Also we recall that [5, Lemma 5.2.2; 5, Lemma 5.2.3; 2, Theorem 1]

(1.4) $\qquad Y$ complete, T onto $\Longrightarrow T$ almost open,

(1.5) $\qquad X$ complete, T almost open $\Longrightarrow T$ open,

and

(1.6) $\quad T^*$ bounded below $\Longleftrightarrow T$ almost open.

2. Preliminaries

If $T \in L(X, Y)$ write $T^\wedge : X \longrightarrow T(X)$ for its 'truncation': thus $T^\wedge$ is automatically onto, and

(2.1) $\qquad T$ one-one $\Longleftrightarrow T^\wedge$ one-one

and

(2.2) $\qquad T$ bounded below $\Longleftrightarrow T^\wedge$ bounded below.

We shall call $T \in L(X, Y)$ *relatively open* if $T^\wedge$ is open, and call T *relatively almost open* if $T^\wedge$ is almost open (cf. [4] (3.5.5)).

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Evidently,

(2.3) T relatively open $\Longrightarrow$ T relatively almost open,

(2.4) T relatively open and one-one $\Longleftrightarrow$ T bounded below,

(2.5) T relatively open and onto $\Longleftrightarrow$ T open,

(2.6) T relatively open, one-one, and onto $\Longleftrightarrow$ T invertible,

and since T is dense if and only if T^* is one-one, by (1.6) and (2.4)

(2.7) T^* relatively open, T dense $\Longleftrightarrow$ T almost open.

EXAMPLE. Let $T: H \longrightarrow H$ be a non-unitary isometric linear operator on a complex Hilbert space H. Then T is relatively open but not open and the Hilbert-adjoint operator T^* of T is relatively open but not bounded below. Consider a unilateral shift operator.

The following lemma is useful in the sequel (see [6] for proof)

LEMMA 2.1. *If* $T \in L(X, Y)$ *then*

T relatively almost open $\Longleftrightarrow$ T^* relatively open.

3. Relatively open spectrum in $L(X)$

In this section we shall consider the spectrum in the context of the Banach algebra $L(X)$, where X is a nonzero complex Banach space. If $T \in L(X)$ write briefly $\sigma(T)$ instead of $\sigma_{L(X)}(T)$, where $\sigma_{L(X)}(T)$ is the spectrum of T relative to $L(X)$. Also we write $\sigma_p(T)$, $\sigma_{com}(T)$, $\sigma_{ap}(T)$, $\sigma_r(T)$, and $\sigma_c(T)$ for the point spectrum, compression spectrum, approximate point spectrum, residual spectrum, and continuous spectrum of $T \in L(X)$, respectively (cf. [1]).

The concepts of "relatively open" and "relatively almost open" invite naming the corresponding subsets of the spectrum:

DEFINITION 3.1. Let X be a normed space and let $T \in L(X)$. The set of all complex numbers λ such that $T-\lambda I$ is not relatively open is call the *relatively open spectrum* of T, denoted $\sigma_{ro}(T)$. The set of all complex numbers λ such that $T-\lambda I$ is not relatively almost open is called the *relatively almost open spectrum* of T, denoted $\sigma_{rao}(T)$.

The preceeding results give the following relations:

(3.1) $\sigma_{rao}(T) \subseteq \sigma_{ro}(T)$,

(3.2) $\sigma_{ro}(T) \cup \sigma_p(T) = \sigma_{ap}(T)$,

(3.3) $\sigma_{rao}(T) = \sigma_{ro}(T^*)$.

The following theorem is easily checked:

THEOREM 3.2. *If* $T \in L(X)$ *for a normed space* X *then*

(a) $\sigma_c(T) \cap \sigma_{ap}(T) \subseteq \sigma_{ro}(T)$

(b) $\sigma_{com}(T) \cup \sigma_{rao}(T) = \sigma_{ap}(T^*)$

Proof. (a) If $\lambda \in \sigma_c(T) \cap \sigma_{ap}(T)$ then $T-\lambda I$ is one-one but not bounded below, therefore by (2.4) $T-\lambda I$ is not relatively open, i.e., $\lambda \in \sigma_{ro}(T)$.

(b) Since, by (1.6) and (2.7), $\sigma_{ap}(T^*) = \sigma_{ro}(T^*) \cup \sigma_{com}(T)$ it is immediate from (3.3).

The next main results are the useful characterizations of relatively open spectrum in the Banach space setting:

THEOREM 3.3 *If* $T \in L(X)$ *for a Banach space* X *then*

(a) $\sigma_c \subseteq \sigma_{rao}(T)$

(b) $\sigma_{ro}(T) - \sigma_{rao}(T) \subseteq \sigma_{com}(T)$

(c) $\sigma_{ro}(T) \subseteq \sigma_{ap}(T) \cap \sigma_{ap}(T^*)$

Proof. (a) If $\lambda \in \sigma_c(T)$ then $T-\lambda I$ is not surjective or $T-\lambda I$ is not relatively open, thus $T-\lambda I$ is not open. Hence by (1.5) $T-\lambda I$ is not almost open. Since $T-\lambda I$ is dense, by (2.7) $T^*-\lambda I$ is not relatively open. By Lemma 2.1 $T-\lambda I$ is not relatively almost open; therefore $\lambda \in \sigma_{rao}(T)$.

(b) If $\lambda \in \sigma_{ro}(T) - \sigma_{rao}(T)$ then $T-\lambda I$ is not relatively open but is relatively almost open. Thus by Lemma 2.1 $T^*-\lambda I$ is relatively open. Then by (2.7) $T-\lambda I$ would be almost open if $T-\lambda I$ were dense. Thus by (1.5) $T-\lambda I$ is open; therefore $T-\lambda I$ is relatively open, which is a contradiction. Hence $T-\lambda I$ is not dense, i.e., $\lambda \in \sigma_{com}(T)$.

(c) If $\lambda \in \sigma_{ro}(T)$ then $T-\lambda I$ is not relatively open. Thus $T-\lambda I$ is not bounded below, i.e., $\lambda \in \sigma_{ap}(T)$. Since X is complete, by (1.5) $T-\lambda I$ is not relatively almost open, thus by Lemma 2.1 $T^*-\lambda I$ is not relatively open. Therefore $T^*-\lambda I$ is not bounded below, i.e., $\lambda \in \sigma_{ap}(T^*)$. Thus we have $\lambda \in \sigma_{ap}(T) \cap \sigma_{ap}(T^*)$.

REMARK. In view of Theorem 3.3 (a), we have

$$\sigma(T) = \sigma_p(T) \cup \sigma_{rao}(T) \cup \sigma_r(T).$$

THEOREM 3.4. *If* $T \in L(X)$ *for a Banach space* X, *then* $T-\lambda I$ *is dense but not onto if and only if* $\lambda \in \lambda_{rao}(T) - \sigma_{com}(T)$.

Proof. Suppose $\lambda \in \sigma_{rao}(T) - \sigma_{com}(T)$. Then $T - \lambda I$ is dense and $T - \lambda I$ is not relatively almost open, thus by Lemma 2.1 $T^* - \lambda I$ is not relatively open. If $T - \lambda I$ were onto then by (1.4) $T - \lambda I$ would be almost open; thus by (2.7) $T^* - \lambda I$ is relatively open, which is a contradiction. Therefore $T - \lambda I$ is not onto.

Conversely, if $T - \lambda I$ is dense but not onto then $\lambda \notin \sigma_{com}(T)$. If $T^* - \lambda I$ were relatively open then by (2.7) $T - \lambda I$ would be almost open. Thus by (1.5) $T - \lambda I$ is open, hence $T - \lambda I$ is onto, which violates our assumption. Therefore $\lambda \in \sigma_{ro}(T^*)$, thus by (3.3) $\lambda \in \sigma_{rao}(T)$. Hence $\lambda \in \sigma_{rao}(T) - \sigma_{com}(T)$.

THEOREM 3.5. *If $T \in L(X)$ for a Banach space X then $T - \lambda I$ is open but not bounded below if and only if $\lambda \in \sigma(T) - \sigma_{ap}(T^*)$.*

Proof. Suppose $\lambda \in \sigma(T) - \sigma_{ap}(T^*)$. Then $T^* - \lambda I$ is bounded below and singular. By (1.6) $T - \lambda I$ is almost open and singular; thus by (1.5) $T - \lambda I$ is open but not one-one. Therefore $T - \lambda I$ is open but not bounded below.

Conversely, if $T - \lambda I$ is open but not bounded below then $T - \lambda I$ is open but not one-one. By (1.6) $T^* - \lambda I$ is bounded below and singular. Therefore $\lambda \in \sigma(T) - \sigma_{ap}(T^*)$.

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Sung Kyun Kwan University
Suwon 440-746, Korea