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JOINT SPATIAL NUMERICAL RANGES OF OPERATORS ON BANACH SPACES

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1. Introduction

Throughout this paper, X will always denote a Banach space over the complex numbers $\boldsymbol{C}$, and $L(X)$ will denote the Banach algebra of all continuous linear operators on X. Operator will always mean continuous linear operator. An n-tuple of operators $T_1, \cdots, T_n$ on X will be denoted by $\hat{T}=(T_1, \cdots, T_n)$. Let $L^n(X)$ be the set of all n-tuples of operators on X. X' will denote the dual space of $X, S(X)$ its unit sphere and $\Pi(X)$ the subset of $X\times X'$ defined by

$$\Pi(X)=\{(x,f)\in X\times X' : \|x\|=\|f\|=f(x)=1\}.$$

For each $T\in L(X)$, we let T^* denote the dual operator. Given a subset B of X, we let $\bar{B}$ denote the closure of B. If $z=(z_1, \cdots, z_n)\in \boldsymbol{C}^n$, we let $|z|=(\sum_{i=1}^{n}|z_i|^2)^{1/2}$.

Let T be an operator on X. It is well known that the spatial numerical range $V(T)$ of T is not closed, not in general convex, but that it is always connected [1]. Also $\overline{V(.)}$ is a continuous mapping of $L(X)$, endowed with norm topology to the set of compact subsets of $\boldsymbol{C}$, endowed with the Hausdorff metric topology. In this paper, we will define joint lower numerical range $LV(\hat{T})$ of $\hat{T}\in L^n(X)$, and study the relation between this concept and joint spatial numerical range $V(\hat{T})$, continuous properties of several mappings related to joint numerical ranges and analogous topological properties (connectedness, closedness etc) for an n-tuple of operators. Finally we will give some problems about joint spatial numerical range.

2. Joint spatial numerical ranges

Let $\hat{T}=(T_1, \cdots, T_n)\in L^n(X)$ be an n-tuple of operators on X. The

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joint spatial numerical range, *joint operator norm* and *joint spatial numerical radius* of $\hat{T}$ respectively are defined by

$$V(\hat{T})=\{(f(T_1x),\cdots,f(T_nx))\in \boldsymbol{C}^n : (x,f)\in \Pi(X)\},$$

$$\|\hat{T}\|=\sup\ \{(\sum_{i=1}^{n}\|T_ix\|^2)^{1/2} : \|x\|=1,\ x\in X\},$$

and

$$\nu(\hat{T})=\sup\ \{\sum_{i=1}^{n}|z_i|^2)^{1/2} : (z_1,\cdots,z_n)\in V(\hat{T})\}$$

respectively ([3], [7]). We also define another joint operator norm of T by $|\hat{T}|=(\sum_{i=1}^{n}\|T_i\|^2)^{1/2}$. Clearly $\nu(\hat{T})\leq\|\hat{T}\|\leq|\hat{T}|$.

Given $x\in S(X)$, let $D(X,x)=\{f\in S(X') : f(x)=1\}$, and $V(\hat{T},x)=\{(f(T_1x),\ \cdots,\ f(T_nx)) : f\in D(X,x)\}$. Clearly by the Hahn-Banach theorem, $V(\hat{T},x)\neq\phi$, $V(\hat{T})=\cup\{V(\hat{T},x) : x\in S(X)\}$, and $V(\hat{T})$ is a nonempty and bounded subset of $\boldsymbol{C}^n$.

Given $a_1,\cdots,a_n$ of a unital normed algebra A, let $V(A\ ;a_1,\cdots,a_n)=\{(f(a_1),\cdots,f(a_n)) : f\in A',\ f(1)=\|f\|=1\}$ denote the joint numerical range of $a_1,\cdots,a_n$ [1]. Clearly $V(A\ ;a_1,\cdots,a_n)$ is a compact convex subset of $\boldsymbol{C}^n$. Also for each n-tuple $\hat{T}=(T_1,\cdots,T_n)$ of operators on X,

$$V(\hat{T})\subset V(B(X)\ ;\hat{T})=V(B(X)\ ;T_1,\cdots,T_n).$$

LEMMA 2.1. *Let $x\in S(X)$ and $\hat{T}=(T_1,\cdots,T_n)\in L^n(X)$ be an n-tuple of operators on X. Then $V(\hat{T},x)$ is the set of all $\lambda=(\lambda_1,\cdots,\lambda_n)\in\boldsymbol{C}^n$ such that for each $i=1,\cdots,n$,*

$$|\lambda_i-\mu_i|<\|(T_i-\mu_iI)x\|\quad(\mu=(\mu_1,\cdots,\mu_n)\in\boldsymbol{C}^n).$$

Proof. The proof is similar to that of Lemma 15.1 [2].

Let $X_{\boldsymbol{R}}$ denote the space X regarded as a Banach space over $\boldsymbol{R}$. Then the mapping $f\longrightarrow \mathrm{Re}\ f$ is an isometric real linear mapping of X' onto $X'_{\boldsymbol{R}}$, and the mapping $(x,f)\longrightarrow(x,\ \mathrm{Re}\ f)$ maps $\Pi(X)$ onto $\Pi(X_{\boldsymbol{R}})$ [2]. From these facts, we obtain the following result.

THEOREM 2.2. *Let $\hat{T}=(T_1,\cdots,T_n)\in L^n(X)$, and $\hat{T}_{\boldsymbol{R}}$ denote $\hat{T}$ regarded as an n-tuple of operators on $X_{\boldsymbol{R}}$. Then*

$$V(\hat{T}_{\boldsymbol{R}})=Re\ V(\hat{T})=\{(Re\,f(T_1x),\cdots,Re\,f(T_nx)) : (x,f)\in\Pi(X)\}.$$

LEMMA 2.3 ([2]). *Let X, Y be metric spaces with Y compact, and let ϕ be a mapping of X into 2^Y such that $\phi(x)$ is closed for each $x\in X$. Then ϕ is upper semi-continuous (u. s. c) if and only if*

$$x_n\in X,\ y_n\in\phi(x_n)\ (n=1,2,\ldots),\ x=\lim_{n\to\infty}x_n,\ y=\lim_{n\to\infty}y_n\ \textit{imply}\ y\in\phi(x).$$

From this lemma and the similar method of proof in Lemma 15.8 [2], we obtain the following result.

THEOREM 2.4. *For each $\hat{T}\in L^n(X)$, the mapping $x\longrightarrow V(\hat{T},x)$ is an upper semi-continuous mapping of $S(X)$ with the norm topology into the nonvoid compact convex subsets of $\boldsymbol{C}^n$.*

$L^n(X)$ becomes an algebra with involution if we define all the operations componentswise. In particular, if $\hat{S}=(S_1,\cdots,S_n)$ and $\hat{T}=(T_1,\cdots,T_n)$ are elements of $L^n(X)$, we have $\hat{S}^*=(S_1{}^*,\cdots,S_n{}^*)$, $\hat{S}\hat{T}=(S_1T_1,\cdots,S_nT_n)$, and a norm is defined by $\|\hat{T}\|=\sup\{(\sum_{i=1}^{n}\|T_ix\|^2)^{1/2} : x\in X,\ \|x\|=1\}$.

THEOREM 2.5. *The mapping $\hat{T}\longrightarrow\overline{V(\hat{T})}$ is a continuous mapping from $L^n(X)$, endowed with a norm topology to the set of compact subsets of $\boldsymbol{C}^n$, endowed with the Hausdorff metric topology. Also $\nu(.)$ is a continuous real-valued mapping on $L^n(X)$.*

Proof. Let $\hat{S}=(S_1,\cdots,S_n)$ and $\hat{T}=(T_1,\cdots,T_n)$ be any elements of $L^n(X)$. If $\|\hat{S}-\hat{T}\|<\varepsilon$ and $(x,f)\in\Pi(X)$, then

$$\begin{aligned}&|(f(S_1x),\cdots,f(S_nx))-(f(T_1x),\cdots,f(T_nx))|\\&=|(f((S_1-T_1)x),\cdots,f((S_n-T_n)x))|\\&=(\sum_{i=1}^{n}|f((S_i-T_i)x)|^2)^{1/2}\le\|\hat{S}-\hat{T}\|,\end{aligned}$$

and so

$$\begin{aligned}(f(S_1x),\cdots,f(S_nx))&=(f(T_1x),\cdots,f(T_nx))+\\&\quad(f((S_1-T_1)x),\cdots,f((S_n-T_n)x))\\&\in V(\hat{T})+\varepsilon.\end{aligned}$$

Thus $\overline{V(\hat{S})}\subset\overline{V(\hat{T})}+(\varepsilon)$. By symmetry, $\overline{V(\hat{T})}\subset\overline{V(\hat{S})}+(\varepsilon)$. Thus $\|\hat{S}-\hat{T}\|<\varepsilon$ implies $d(\overline{V(\hat{S})},\ \overline{V(\hat{T})})\le\varepsilon$, and $\overline{V(.)}$ is a continuous mapping from $L^n(X)$ to the set of compact subsets of $\boldsymbol{C}^n$, endowed with the Hausdorff metric topology.

Also $\nu(\hat{S})\le\nu(\hat{T})+\varepsilon$ and $\nu(\hat{T})\le\nu(\hat{S})+\varepsilon$ imply $|\nu(\hat{S})-\nu(\hat{T})|\le\varepsilon$. So $\nu(.)$ is a continuous real-valued mapping on $L^n(X)$.

Given an n-tuple $\hat{T}=(T_1,\cdots,T_n)$ of operators on X', *the joint lower numerical range* $LV(\hat{T})$ of $\hat{T}$ is defined by

$$LV(\hat{T})=\{((T_1f)x,\ \cdots,(T_nf)x) : (x,f)\in\Pi(X)\}.$$

THEOREM 2.6. *Let* $\hat{T}=(T_1,\cdots,T_n)$ *be an n-tuple of operators on* X' (*not necessarily commuting!*). *Then*

$$LV(\hat{T})\subset V(\hat{T})\subset\overline{LV(\hat{T})}.$$

Proof. Let $\hat{x}$ denote the canonical image in X'' of $x\in X$. Then $(x,f)\in\Pi(X)$ implies $(f,\hat{x})\in\Pi(X')$, and so $(T_jf)(x)=\hat{x}(T_jf)$ $(j=1,\cdots,n)$. Hence $LV(\hat{T})\subset V(\hat{T})$.

Let $\lambda=(\lambda_1,\cdots,\lambda_n)\in V(\hat{T})$ i.e., for each $j=1,\cdots,n$, $\lambda_j=\phi(T_jf)$ with $(f,\phi)\in\Pi(X')$. Let ε be an arbitrary number with $0<\varepsilon<1$, and let X_1 denote the closed unit ball of X. Since $\hat{X}_1$ is weak*-dense in X_1'', there exist $x\in X_1$ with

$$|\phi(f)-\hat{x}(f)|<(\varepsilon/2)^2,\quad |\phi(T_jf)-\hat{x}(T_jf)|<\varepsilon/\sqrt{n}\quad (j=1,\cdots,n).$$

We have $x\in X_1, f\in S(X')$, and $|1-f(x)|<(\varepsilon/2)^2$. Therefore by the Bishop-Phelps-Bollobás theorem [2], there exists $(y,g)\in\Pi(X)$ such that $\|y-x\|<\varepsilon$, $\|f-g\|<\varepsilon$. Then $(\hat{T}g)(y)=((T_1g)y,\cdots,(T_ng)y)\in LV(\hat{T})$ and

$$\begin{aligned}|\lambda-(\hat{T}g)(y)|&\le|\lambda-\hat{x}(\hat{T}f)|+|(\hat{T}f)(x)-(\hat{T}f)(y)|\\&\quad+|(\hat{T}f)(y)-(\hat{T}g)(y)|\\&<\varepsilon+|\hat{T}|\,\|x-y\|+|\hat{T}|\,\|f-g\|\\&<\varepsilon(1+2|\hat{T}|).\end{aligned}$$

Since ε is arbitrary, $\lambda\in\overline{LV(\hat{T})}$.

COROLLARY 2.6. *For each* $\hat{T}=(T_1,\cdots,T_n)\in L^n(X)$, $V(\hat{T})\subset V(\hat{T}^*)\subset\overline{V(\hat{T})}$.

Proof. This follows from the above theorem and the fact that $LV(\hat{T}^*)=V(\hat{T})$.

The following example shows that we can have $V(\hat{T})\neq V(\hat{T}^*)$.

EXAMPLE 2.7. Let $X=c_0$ and define $T\in B(X)$ by

$(Tx)(n)=\sum_{k=0}^{\infty}2^{-k-1}x(n+k)$ $(n=1,2,3,\cdots)$. Then $V(T,T)\neq V(T^*,T^*)$ i.e., $V(\hat{T})\neq V(\hat{T}^*)$ for $\hat{T}=(T,T)$.

Proof. It is clear that $\|Tx\|<1$ for $\|x\|=1$, and so $(1,1)\notin V(\hat{T})$. With the normal identifications, $X'=l^1$, $X''=l^\infty$, and $(T^*f)(n)=\sum_{k=1}^{n}2^{k-n-1}f(k)$ $(n=1,2,\ldots)$. Let $f\in X'$, $\phi\in X''$ be defined by $f(n)=\delta_n{}^1$, $\phi(n)=1$ $(n=1,2,\ldots)$. Then $(f,\phi)\in\Pi(X')$ and $(1,1)=(\phi(T^*f),\phi(T^*f))\in V(\hat{T}^*)$.

Let $(X, \|\cdot\|)$ denote a complex Banach space and let $N(X)$ denote the set of all norms on X equivalent to $\|\cdot\|$. Given $p, q \in N(X)$, we define $\mu(p, q)$, $d(p, q)$ by

$$\mu(p, q) = \inf\{\delta \geq 1 : 1/\delta \leq p(x)/q(x) \leq \delta,\ x \in X - \{0\}\},$$
$$d(p, q) = \log \mu(p, q).$$

We note that d is a metric on $N(X)$, and $(N(X), d)$ is complete [2].

From now on, it is convenient to denote also by p the dual norm of p. The set Π_p is then defined by

$\Pi_p = \{(x, f) \in X \times X' : p(x) = p(f) = f(x) = 1\}$, and given $\hat{T} = (T_1, \cdots, T_n) \in L^n(X)$, $V_p(\hat{T}) = \{(f(T_1x), \cdots, f(T_nx)) : (x, f) \in \Pi_p\}$.

THEOREM 2.8. *The mapping* $p \longrightarrow \overline{V_p(\hat{T})}$ *is a continuous mapping from* $N(X)$, *endowed with the above defined metric* d *to the set of compact subsets of* $\boldsymbol{C}^n$, *endowed with the Hausdorff metric topology.*

Proof. With $\kappa > 1$, we let $G_\kappa = \{p \in N(X) : \mu(p, \|\cdot\|) < \kappa\}$. Then by Theorem 18.3 [2], $p \longrightarrow \Pi_p$ is uniformly continuous on G_κ. Given $\varepsilon > 0$, choose $\delta > 0$ such that $p, q \in G_\kappa$, $d(p, q) < \delta$ implies $d(\Pi_p, \Pi_q) < \varepsilon$. Given $\lambda = (\lambda_1, \cdots, \lambda_n) \in V_q(\hat{T})$, we have $\lambda = (f(T_1x), \cdots, f(T_nx))$ for some $(x, f) \in \Pi_q$. Then $d((x, f), \Pi_p) < \varepsilon$ and so there exists $(y, g) \in \Pi_p$ with $\|x - y\| + \|f - g\| < \varepsilon$. Then

$$\begin{aligned} &|(g(T_1y), \ldots, g(T_ny)) - (f(T_1x), \ldots, f(T_nx))| \\ &\leq |(g(T_1y), \ldots, g(T_ny)) - (f(T_1y), \ldots, f(T_ny))| \\ &+ |(f(T_1y), \ldots, f(T_ny)) - (f(T_1x), \ldots, f(T_nx))| \\ &\leq \|g - f\| |\hat{T}| \|y\| + \|f\| |\hat{T}| \|y - x\| \\ &\leq \|g - f\| |\hat{T}| \kappa + \kappa |\hat{T}| \|y - x\| < \varepsilon\kappa |\hat{T}|. \end{aligned}$$

Thus $\sup\{d(\lambda, \overline{V_p(\hat{T})}) : \lambda \in V_q(\hat{T})\} \leq \varepsilon\kappa |\hat{T}|$, and by symmetry, $\sup\{d(\lambda, \overline{V_q(\hat{T})}) : \lambda \in V_p(\hat{T})\} \leq \varepsilon\kappa |\hat{T}|$. Hence

$$d(\overline{V_q(\hat{T})}, \overline{V_p(\hat{T})}) \leq \varepsilon\kappa |\hat{T}|.$$

3. Topological properties of joint numerical ranges

For every $T \in L(X)$, it is well-known that the spatial numerical range $V(T)$ of T is not in general convex, not closed, but that it is always connected. The norm$\times$weak* topology is defined as the product topology on $X \times X'$ given by the norm topology on X and the weak* topology on X' [1].

LEMMA 3.1. *Let* π_1 *denote the natural projection* $\pi_1(x, f) = x$ *of* $X \times X'$

onto X, *and for each* $p\in N(X)$, *let* E *be a subset of* Π_p *that is relatively closed in* Π_p *with respect to the norm*$\times$*weak* topology. Then* $\pi_1(E)$ *is a norm closed subset of* X.

THEOREM 3.2. *For each* $p\in N(X)$, Π_p *is a connected subset of* $X\times X'$ *with the norm*$\times$*weak* topology unless* X *has dimension one over* $\boldsymbol{R}$.

Proof. The proof is similar to that of Theorem 11.4 [1].

COROLLARY 3.3. *For each* $p\in N(X)$, $V_p(\hat{T})$ *is connected.*

Proof. We have

$$|f(T_ix)-g(T_iy)|\leq p(T_ix-T_iy)+|(f-g)(T_iy)|$$

$((x,f),(y,g)\in\Pi_p,\ i=1,2,\dots,n)$. Therefore the mapping $(x,f)\longrightarrow f(T_ix)$ is a continuous mapping of Π_p with the relative norm$\times$weak* topology onto $V_p(T_i)$. Since the natural projection $\pi_i:\boldsymbol{C}^n\longrightarrow\boldsymbol{C}$ is continuous, the mapping $(x,f)\longrightarrow(f(T_1x),\dots,f(T_nx))$ is a continuous mapping of Π_p with the relative norm$\times$weak* topology onto $V(\hat{T})$. By the above theorem, $V(\hat{T})$ is connected, except perhaps when X has dimension one over $\boldsymbol{R}$.

Finally let X have dimension one over $\boldsymbol{R}$, and take $u\in X$ with $p(u)=1$. Then every $x\in X$ is of the form $x=\zeta u$ with $p(x)=|\zeta|$, and for every $f\in X'$, we have $f(x)=\zeta f(u)$, $p(f)=|f(u)|$. Let g be the functional given by $g(x)=\zeta$ $(x=\zeta u\in X)$. Then $S(X)=\{u,-u\}$, and Π_p has exactly two points (u,g) and $(-u,\ -g)$, and so $V(\hat{T})$ has exactly one point $(g(T_1u),\dots,g(T_nu))$.

COROLLARY 3.4. *Let* $\hat{F}=(F_1,\dots,F_n)$ *be an* n*-tuple of continuous mappings of* $S(X)$ *into* X, *and for each* $p\in N(X)$, *let* $V_p(\hat{F})=\{(f(F_1x),\dots,f(F_nx)):(x,f)\in\Pi_p\}$. *Then* $V_p(\hat{F})$ *is connected, except perhaps when* X *has dimension one over* $\boldsymbol{R}$.

For each n-tuple $\hat{T}=(T_1,\dots,T_n)$ of operators on X, the *joint numerical range* $W(\hat{T})$ of $\hat{T}$ is given by

$$W(\hat{T})=\{([T_1x,x],\dots,[T_nx,x]):\|x\|=1\},$$

where $[\,,]$ is a consistent semi-inner product (s. i. p) of Lumer [6]. It is easy to see that $W(\hat{T})\subset V(\hat{T})\subset V(B(X);\hat{T})$, and $V(\hat{T})$ is the union of all numerical ranges $W(\hat{T})$ corresponding to all consistent s. i. p's on X. However, $V(\hat{T})=W(\hat{T})$ in case of smooth space.

From the following example, we see that $W(\hat{T})$ is disconnected.

EXAMPLE 3.5. Let $X=\boldsymbol{C}^2$ with the sup norm $\|x\|=\max(|\zeta_1|, |\zeta_2|)$ $(x=(\zeta_1, \zeta_2)\in X)$, and let $[x, y]$ be defined for $x=(\zeta_1, \zeta_2)$, $y=(\eta_1, \eta_2)$, by

$$[x, y]=\begin{cases}\zeta_1 \ \eta_1{}^* & (\text{if } |\eta_1|=\|y\|),\\ \zeta_2 \ \eta_2{}^* & (\text{if } |\eta_1|<\|y\|).\end{cases}$$

Then $[\ ,\ ]$ is a s.i.p on X, and $[x, x]=\|x\|^2$. Define the operator T_j by $T_jx=(\zeta_1, 0)$ $(x=(\zeta_1, \zeta_2)\in X,\ j=1, 2)$. For each 2-tuple $\hat{T}=(T_1, T_2)$, we have

$W(\hat{T}) = W(T_1, T_2)=\{(0, 0), (1, 1)\}$. For if $\|x\|=1$ and $|\zeta_1|=1$, then $([T_1x, x], [T_2x, x])=(\zeta_1\zeta_1{}^*, \zeta_1\zeta_1{}^*)=(1, 1)$, and if $\|x\|=1$ and $|\zeta_1|<1$, then

$$([T_1x, x], [T_2x, x])=(0\zeta_2{}^*, 0\zeta_2{}^*)=(0, 0).$$

Thus $W(T)=\{(0, 0), (1, 1)\}$ is disconnected.

The following example shows that $V(\hat{T})$ is not closed as well as $V(T)$ in single operator case.

EXAMPLE 3.6. Let $X=l^2$ and let S be defined by $S(x_1, x_2, x_3, \dots)=(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$. We consider the tensor product of operators T_j $(j=1, 2)$ on the tensor product $X\otimes X$ defined by $T_1=S\otimes I$, and $T_2=I\otimes S$. Then $V(\hat{T})=W(T_1, T_2)=W(S)\times W(S)=(0, 1]\times(0, 1]$, and so $V(\hat{T})$ is not closed.

Now we have the following problems.

REMARKS 3.7. (a) For what Banach space X and $\hat{T}\in L^n(X)$ is $V(\hat{T})$ a closed set?

(b) Let K be a compact simply connected subset of $\boldsymbol{C}^n$. Is there some space X and $\hat{T}\in L^n(X)$ such that $V(\hat{T})=K$?

(c) For what Banach space X is $V(\hat{T})$ convex for every $\hat{T}\in L^n(X)$?

(d) Is $V(\hat{T})$ always simply connected for every $\hat{T}\in L^n(X)$?

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