BRAUER GROUP OVER A KRULL DOMAIN

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Let $R$ be a Krull domain with field of fractions $K$. By $\text{Br}(R)$ we denote the Brauer group of $R$. Studying the Kernel of the homomorphism $\text{Br}(R) \to \text{Br}(K)$, Orzech defined Brauer groups $\text{Br}(M)$ for different categories $M$ of $R$–modules [4].

In this paper we show that an algebra $A$ in $\text{Br}(D)$ is a maximal order in $A \otimes K$ and that the map $\text{Br}(D) \to \text{Br}(K)$ is one to one.

We note here few conventions. All rings are Krull domains and all modules will be unitary. By $Z$ we donote the set of height one prime ideals of a Krull domain.

0. Preliminaries

We first recall the following definitions and basic properties taken from [4].

(1) An $R$–module $M$ is divisorial if it is torsion free and in $K \otimes M$ the equality $M = \bigcap_{p \in Z} M_p$ holds.

(2) An $R$–module $M$ is an $R$–lattice if $M$ is torsion free of finite rank and there exists an $R$–module $F$ of finite type such that $M \subseteq F \subseteq M \otimes K$.

Let $D$ be the category of divisorial $R$–lattices. For $M$ and $N$ in $D$, we view $M \otimes N$ as a subset of $(M \otimes_R K) \otimes_K (N \otimes_R K)$ and define

$$M \_N := \bigcap_{p \in Z} (M \otimes N)_p$$

Let $Az(D)$ be the set of isomorphism classes of central $R$–algebras $A$ which are in $D$ as $R$–modules, and for which the following natural map $\gamma_A : A \_ A^0 \to \text{End}_R(A)$ induced by the map $A \otimes A^0 \to \text{End}_R(A)$ is an isomorphism. We note that $R$–algebra $A$ is in $Az(D)$ if and only if $A$ is a divisorial $R$–lattice and $A_P$ is an $R_P$–Azumaya algebra (i.e. $A_P$ is a central separable $R_P$–algebra) for all $p$ in $Z$.

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We define an equivalence relation \( \sim \) on \( \text{Az}(D) \) by
\[
A \sim B \quad \text{if} \quad A \bot \text{End}_R(M) = B \bot \text{End}_R(N)
\]
for some \( M \) and \( N \) in \( D \). Let \( \text{Br}(D) \) denote the set of equivalence classes of \( \text{Az}(D) \) and let \( [A] \) denote the class of \( A \). Then \( \text{Br}(D) \) is an abelian group under the operation \( [A][B] = [A \bot B] \). The identity element in this group is given by \( [\text{End}_R(E)] \) for some \( E \in D \) and \( [A]^{-1} = [A^0] \).

Since every faithfully projective \( R \)-module is in \( D \), there is a group homomorphism \( \text{Br}(R) \to \text{Br}(D) \). For any \( R \)-algebra \( A \) and any prime ideal \( \mathfrak{p} \) of \( R \), \( A \otimes R \mathfrak{p} \cong A_{\mathfrak{p}} \otimes R_{\mathfrak{p}} \mathfrak{p} \), we have the induced group homomorphism \( \text{Br}(D) \to \text{Br}(K) \). In [4] Orzech proved that the kernel of the map \( \text{Br}(R) \to \text{Br}(K) \) is exactly the kernel of the map \( \text{Br}(R) \to \text{Br}(D) \).

1. Main Theorem

Let \( R \) be a regular domain and let \( R \)-algebra \( A \) be an Azumaya algebra. Then it is well known that \( A \) is a maximal \( R \)-order in \( A \otimes K \) [3]. Similarly the following holds:

**Proposition 1.** Let \( R \)-algebra \( A \) be a divisorial \( R \)-lattice such that \( A_{\mathfrak{p}} \) is a central separable \( R_{\mathfrak{p}} \)-algebra for every \( \mathfrak{p} \in \mathfrak{Z} \) (i.e. \( [A] \in \text{Br}(D(R)) \)). Then \( A \) is a maximal \( R \)-order in \( K \otimes A \).

**Proof.** Let \( B \) be the integral closure of \( R \) in \( A \). Since \( B \) contains a \( K \)-basis of \( K \otimes A \) which is in \( A, B \) is an \( R \)-order in \( K \otimes A \). For each \( \mathfrak{p} \in \mathfrak{Z}, A_{\mathfrak{p}} \) is an (maximal) \( R \)-order in \( K \otimes A_{\mathfrak{p}} \) by proposition 6.18 [3] and hence \( B_{\mathfrak{p}} = A_{\mathfrak{p}} \). By proposition 6.11 [3] \( B^{**} \) is an \( R \)-order in \( K \otimes A \). From the following canonical inclusions
\[
B \subset B^{**} \subset A
\]
and \( B_{\mathfrak{p}} = A_{\mathfrak{p}} \), we have \( i_{\mathfrak{p}} : B^{**} \otimes R_{\mathfrak{p}} = A_{\mathfrak{p}} \). Since \( B^{**} \) and \( A \) are divisorial (equivalently reflexive) \( R \)-modules, \( i : B^{**} \to A \) is an isomorphism by Lemma 1.1 [4] and hence \( A \) is an \( R \)-order in \( K \otimes A \). Since \( A_{\mathfrak{p}} \) is a maximal \( R \)-order in \( K \otimes A_{\mathfrak{p}} \) for each \( \mathfrak{p} \in \mathfrak{Z} \), by proposition 1.3 [1], \( A \) is a maximal \( R \)-order in \( A \otimes K \).

**Theorem 2.** The map \( \text{Br}(D) \to \text{Br}(K) \) is a monomorphism.

**Proof.** Let \( [A] \) be in \( \text{Br}(D) \) which becomes trivial in \( \text{Br}(K) \). Then there is a finite dimensional vector space \( V \) over \( K \) such that \( A \otimes K \cong \text{Hom}_K(V, V) \). By Proposition 1, \( A \) is a maximal \( R \)-order in
the central simple $K$-algebra $\text{Hom}_K(V, V)$. By Proposition 1.7 [1], $A \approx \text{End}_R(E)$ for some divisorial $R$-lattice $E$. By the definition of $\text{Br}(D)$, $[[\text{End}_R(E)] = [A]$ is the identity element in $\text{Br}(D)$ and hence the map is a monomorphism.

**Corollary.** $\text{Br}(D)$ is a torsion group.

**References**


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